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Scaling limits for wave pulse transmission and reflection operators

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The random paraxial wave equation is revisited to take into account not only random forward scattering, but also random backscattering. In this paper we are interested in the transmitted wave fronts and also wave fronts reflected by a strong interface buried in a random medium. In the weakly heterogeneous regime the reflected and transmitted wave fields are characterized by reflection and transmission operators that are the solutions of Itô–Schrödinger diffusion models. These models allow for the computations of the Wigner distributions and the autocorrelation functions of the reflected and transmitted waves. They also fully take into account the fact that the waves travel through the same medium during the propagation to and from the interface, which induces an increase of the beam radius and of the correlation radius, and also predict the enhanced backscattering effect in the backscattered direction.

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1. Introduction

Random wave propagation in the paraxial regime is a well-known model that is used in many applications in communication and imaging [12]. It provides a simple tool for computing the wave transmission through a random medium by taking into account random forward scattering and by neglecting random backscattering. This enables one to study a wide range of phenomena, such as laser beam spreading [5,17] and time reversal in random media [1,3,6]. In many situations of interest, such as in optical coherence tomography and in geophysical imaging, the quantity of interest is the wave reflected by an interface located in the random medium. The usual approach found in most papers is to apply the paraxial wave equation in both directions of propagation and to assume that the statistics of the forward- and backward-propagating waves are independent [18–20]. However, some authors have already noticed that the correlation of the forward–backward propagating events can induce modifications of the autocorrelation function of the reflected wave [13,14]. In our paper we analyze the full statistical distribution of the transmitted and reflected waves. We characterize them by reflection and transmission operators that are the solutions of the Itô–Schrödinger diffusion models (19) and (33) and which take into account forward scattering and backscattering by the random medium and backscattering by the interface. These models enable us to compute low- and high-order moments of the reflected and transmitted wave fields. In particular we obtain closed-form expressions for the first-order and second-order statistics (the coherent wave and the autocorrelation function or Wigner distribution) in several physically relevant regimes.

A particular application of these results is the identification of the regimes in which the independent approach is valid. In the regime in which the transverse correlation length of the medium is smaller than the beam width, we will show that the independent approach gives the correct answer as far as the reflected intensity profile and the spatial autocorrelation function are concerned. However, an important phenomenon not captured by the independent approach, and captured by the

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Itô–Schrödinger diffusion model (33), is the enhanced backscattering or weak localization effect [2,16]: if a quasi-plane wave is incoming with a given incidence angle, then the mean reflected intensity has a local maximum in the backscattered direction, which is twice as large as the mean reflected intensity in the other directions. This enhancement can be observed in a small cone around the backscattered direction, and it can be interpreted as the result of constructive interferences between reciprocal wave paths. On the other hand, in the regime in which the transverse correlation length of the medium is larger than the beam width, the independent approach gives a qualitatively wrong prediction. In particular, it underestimates the beam spreading and it also predicts a correlation radius for the reflected wave smaller than the one that is actually obtained. In fact, the correlation radius of the reflected wave at the surface can be larger than the correlation radius of the wave at the reflecting interface, meaning that the wave recovers part of its coherence when it propagates back in the same random medium. The full Itô–Schrödinger diffusion model that we develop in this paper is needed to explain this phenomenon.

The outline of the paper is as follows: We define the transmission and reflection operators and express the transmitted and reflected wave fields in terms of these operators in Section 2. In Section 3, respectively, 4, we develop a diffusion formulation for the transmission, respectively, reflection, operator. We analyze the Wigner distributions associated with these operators in Sections 5 and 6. This analysis is then used to study quantitatively the wave transmission and reflection in Sections 7 and 8.

2. The transmission and reflection operators

We consider linear acoustic waves propagating in $1+d$ spatial dimensions with heterogeneous and random medium fluctuations. The governing equations are

$$\rho(z,\mathbf{x})\frac{\partial \mathbf{u}}{\partial t} + \nabla p = \mathbf{F}, \quad \frac{1}{K(z,\mathbf{x})}\frac{\partial p}{\partial t} + \nabla\cdot\mathbf{u} = 0, \tag{1}$$

where p is the pressure, $\mathbf{u}$ is the velocity, ρ is the density of the medium, K is the bulk modulus of the medium, and $(z,\mathbf{x}) \in \mathbb{R}\times\mathbb{R}^d$ are the space coordinates. The source is modeled by the forcing term $\mathbf{F}$. We consider in this paper the situation in which a random slab occupying the interval $z \in (0,L)$ is sandwiched between two homogeneous half-spaces. The source, $\mathbf{F}$, is located outside of the slab, in the halfspace $z > L$. We shall refer to waves propagating in a direction with a positive z component as right-propagating waves. The medium fluctuations in the random slab $(0,L)$ vary relatively rapidly in space while the "background" medium is constant. We normalize the background bulk modulus $\overline{K}$ and density $\bar{\rho}$ in the slab to one, so that the background speed $\bar{c} = \sqrt{\overline{K}/\bar{\rho}}$ and impedance $\overline{Z} = \sqrt{\overline{K}\bar{\rho}}$ are also equal to one. The medium is assumed to be matched at the right boundary $z = L$. We consider a possible mismatch at the boundary $z = 0$ and denote the medium parameters in the half-space $z < 0$ by ρ_0 and K_0:

$$\frac{1}{K(z,\mathbf{x})} = \begin{cases} K_0^{-1} & \text{if } z \leqslant 0, \\ 1+\varepsilon\nu(z/\varepsilon^2,\mathbf{x}/\varepsilon) & \text{if } z\in(0,L), \\ 1 & \text{if } z \geqslant L, \end{cases} \qquad \rho(z,\mathbf{x}) = \begin{cases} \rho_0 & \text{if } z \leqslant 0, \\ 1 & \text{if } z\in(0,L), \\ 1 & \text{if } z\geqslant L, \end{cases}$$

with ε a small parameter. The random field $\nu(z,\mathbf{x})$ models the medium fluctuations and we assume that it is stationary and that it satisfies strong mixing conditions. We consider a scaling where the central wavelength of the source is of order ε^2 and write

$$\mathbf{F}(t,z,\mathbf{x}) = f\left(\frac{t}{\varepsilon^2},\frac{\mathbf{x}}{\varepsilon}\right)\delta(z-z_0)\mathbf{e}_z, \tag{2}$$

where $\mathbf{e}_z$ is the unit vector pointing in the z-direction and $z_0 > L$. Note that:

- The source has been normalized so that the Rayleigh length is of order one. The Rayleigh length is the distance from beam waist where the beam area is doubled by diffraction. For a beam with carrier wavenumber k_0 and radius r_0 it is of the order of $k_0 r_0^2$. Here the carrier wavenumber is of order ε^{-2} and the beam radius is of order ε. Therefore the Rayleigh length is of order one.
- The transverse and longitudinal scales of variation of the random medium, ε and ε^2, respectively, correspond to the scales of the waves. Therefore the random scattering is sensitive to the full $1+d$-dimensional spectrum of the random fluctuations (or its $1+d$-dimensional autocorrelation function).
- The amplitude ε of the relative medium perturbation is chosen so that the random effects are significant after a propagation distance of order one. Therefore diffractive and random effects have a nontrivial interplay.

In the companion paper [10] we analyze a different regime, in which the random medium has weak random *isotropic* fluctuations. The techniques used to analyze the wave reflection and transmission are similar in the two papers, however, lead to qualitatively different results. In the regime discussed in [10] bulk random backscattering is asymptotically negligible and the limit Schrödinger system satisfies a conservation of energy relation and describes completely the transmitted and reflected waves. Here bulk random backscattering is of order one and small amplitude, but long, incoherent wave fluctuations are generated. As we will see, the result for the front waves has the form of a dissipative Schrödinger system.

From now on we re-scale $\mathbf{x}/\varepsilon \to \mathbf{x}$ and set

$$p^\varepsilon(t,z,\mathbf{x}) = p(t,z,\varepsilon\mathbf{x}). \tag{3}$$

Therefore, when we refer to the transversal spatial parameter $\mathbf{x}$ in the following, it corresponds to $\varepsilon\mathbf{x}$ in the original coordinates.

The wave field in the homogeneous half-space $z \leqslant 0$ satisfies the wave equation with the wave speed $c_0 = \sqrt{K_0/\rho_0}$. We introduce the complex amplitudes $\check{a}_0^\varepsilon$ and $\check{b}_0^\varepsilon$ of the right- and left-propagating modes:

$$\check{a}_0^\varepsilon(k,z,\mathbf{x}) = \frac{c_0}{2}\left[\int \left(\frac{1}{\varepsilon^2}p^\varepsilon(t,z,\mathbf{x}) + \frac{1}{ik}\frac{\partial p^\varepsilon}{\partial z}(t,z,\mathbf{x})\right)e^{ic_0kt/\varepsilon^2}\,dt\right]e^{-ikz/\varepsilon^2},$$

$$\check{b}_0^\varepsilon(k,z,\mathbf{x}) = \frac{c_0}{2}\left[\int \left(\frac{1}{\varepsilon^2}p^\varepsilon(t,z,\mathbf{x}) - \frac{1}{ik}\frac{\partial p^\varepsilon}{\partial z}(t,z,\mathbf{x})\right)e^{ic_0kt/\varepsilon^2}\,dt\right]e^{ikz/\varepsilon^2}.$$

They are defined such that the pressure field in the region $z \leqslant 0$ can be written as

$$p^\varepsilon(t,z,\mathbf{x}) = \frac{1}{2\pi}\int \left(\check{a}_0^\varepsilon(k,z,\mathbf{x})e^{ikz/\varepsilon^2} + \check{b}_0^\varepsilon(k,z,\mathbf{x})e^{-ikz/\varepsilon^2}\right)e^{-ic_0kt/\varepsilon^2}\,dk,$$

and they satisfy

$$\frac{\partial \check{a}_0^\varepsilon}{\partial z}(k,z,\mathbf{x})e^{ikz/\varepsilon^2} + \frac{\partial \check{b}_0^\varepsilon}{\partial z}(k,z,\mathbf{x})e^{-ikz/\varepsilon^2} = 0.$$

Using (1), we find that they also satisfy the coupled mode equations

$$\frac{\partial \check{a}_0^\varepsilon}{\partial z} = \frac{i}{2k}\Delta_{\mathbf{x}}\check{a}_0^\varepsilon + e^{\frac{-2ikz}{\varepsilon^2}}\frac{i}{2k}\Delta_{\mathbf{x}}\check{b}_0^\varepsilon, \quad \frac{\partial \check{b}_0^\varepsilon}{\partial z} = -e^{\frac{2ikz}{\varepsilon^2}}\frac{i}{2k}\Delta_{\mathbf{x}}\check{a}_0^\varepsilon - \frac{i}{2k}\Delta_{\mathbf{x}}\check{b}_0^\varepsilon,$$

where $\Delta_{\mathbf{x}}$ is the transverse Laplacian. In the limit $\varepsilon \to 0$, the cross terms (proportional to $\exp(\pm 2ikz/\varepsilon^2)$) average out to zero and we get the two uncoupled paraxial wave equations

$$\frac{\partial \check{a}_0^\varepsilon}{\partial z} = \frac{i}{2k}\Delta_{\mathbf{x}}\check{a}_0^\varepsilon, \quad \frac{\partial \check{b}_0^\varepsilon}{\partial z} = -\frac{i}{2k}\Delta_{\mathbf{x}}\check{b}_0^\varepsilon.$$

Taking into account the fact that there is no source in the half-space $z < 0$, and therefore no right-going wave, we obtain

$$p^\varepsilon(t,z,\mathbf{x}) = \frac{1}{2\pi}\int \check{b}_0^\varepsilon(k,z,\mathbf{x})e^{-ikz/\varepsilon^2}e^{-ic_0kt/\varepsilon^2}\,dk, \quad z \leqslant 0. \tag{4}$$

Similarly, the wave field in the homogeneous regions $[L,z_0)$ and (z_0,∞) has the form

$$p^\varepsilon(t,z,\mathbf{x}) = \begin{cases} \frac{1}{2\pi}\int \check{a}_2^\varepsilon(k,z,\mathbf{x})e^{ikz/\varepsilon^2}e^{-ikt/\varepsilon^2}\,dk, & z > z_0, \\ \frac{1}{2\pi}\int \left(\check{a}_1^\varepsilon(k,z,\mathbf{x})e^{ikz/\varepsilon^2} + \check{b}_1^\varepsilon(k,z,\mathbf{x})e^{-ikz/\varepsilon^2}\right)e^{-ikt/\varepsilon^2}\,dk, & z \in [L,z_0). \end{cases}$$

Here we have used the fact that there is no source and therefore no left-going wave in the region $z > z_0$ (see Fig. 1). We can also use the jump conditions across the source position $z = z_0$ to obtain the relations

$$\check{b}_1^\varepsilon(k,z_0,\mathbf{x}) = -\frac{1}{2}e^{ikz_0/\varepsilon^2}\check{f}(k,\mathbf{x}), \quad \check{a}_2^\varepsilon(k,z_0,\mathbf{x}) - \check{a}_1^\varepsilon(k,z_0,\mathbf{x}) = \frac{1}{2}e^{-ikz_0/\varepsilon^2}\check{f}(k,\mathbf{x}).$$

By solving the paraxial wave equation for $\check{b}_1^\varepsilon$, we obtain the expression for the complex amplitude of the wave incoming in the random slab at $z = L$:

$$\check{b}_1^\varepsilon(k,L,\mathbf{x}) = e^{ikz_0/\varepsilon^2}\check{b}_{\mathrm{inc}}(k,\mathbf{x}), \quad \check{b}_{\mathrm{inc}}(k,\mathbf{x}) = -\frac{1}{2(2\pi)^d}\int \hat{f}(k,\boldsymbol{\kappa})e^{\frac{i}{2k}|\boldsymbol{\kappa}|^2(L-z_0)+i\boldsymbol{\kappa}\cdot\mathbf{x}}\,d\boldsymbol{\kappa}, \tag{5}$$

where the transverse spatial Fourier transform is defined by

$$\hat{f}(k,\boldsymbol{\kappa}) = \int \check{f}(k,\mathbf{x})e^{-i\boldsymbol{\kappa}\cdot\mathbf{x}}\,d\mathbf{x}. \tag{6}$$

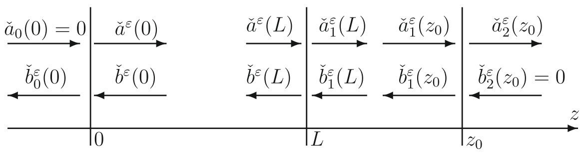

Fig. 1. Boundary conditions for the modes in the presence of an interface at $z = 0$, a random slab $(0,L)$, and a source at $z = z_0$.

The pressure field in the region $z \in (0, L)$ can be written as:

$$p^\varepsilon(t,z,\mathbf{x}) = \frac{1}{2\pi}\int \left(\check{a}^\varepsilon(k,z,\mathbf{x})e^{ikz/\varepsilon^2} + \check{b}^\varepsilon(k,z,\mathbf{x})e^{-ikz/\varepsilon^2}\right)e^{-ikt/\varepsilon^2}\,dk,$$

where the complex amplitudes $\check{a}^\varepsilon$ and $\check{b}^\varepsilon$ of the right- and left-propagating modes are given explicitly by

$$\check{a}^\varepsilon(k,z,\mathbf{x}) = \frac{1}{2}\left[\int\left(\frac{1}{\varepsilon^2}p^\varepsilon(t,z,\mathbf{x}) + \frac{1}{ik}\frac{\partial p^\varepsilon}{\partial z}(t,z,\mathbf{x})\right)e^{ikt/\varepsilon^2}\,dt\right]e^{-ikz/\varepsilon^2},$$

$$\check{b}^\varepsilon(k,z,\mathbf{x}) = \frac{1}{2}\left[\int\left(\frac{1}{\varepsilon^2}p^\varepsilon(t,z,\mathbf{x}) - \frac{1}{ik}\frac{\partial p^\varepsilon}{\partial z}(t,z,\mathbf{x})\right)e^{ikt/\varepsilon^2}\,dt\right]e^{ikz/\varepsilon^2}.$$

We obtain the following coupled mode equations for the complex amplitudes $\check{a}^\varepsilon$ and $\check{b}^\varepsilon$:

$$\frac{\partial \check{a}^\varepsilon}{\partial z} = \left[\frac{ik}{2\varepsilon}v\left(\frac{z}{\varepsilon^2},\mathbf{x}\right) + \frac{i}{2k}\Delta_{\mathbf{x}}\right]\check{a}^\varepsilon + e^{\frac{-2ikz}{\varepsilon^2}}\left[\frac{ik}{2\varepsilon}v\left(\frac{z}{\varepsilon^2},\mathbf{x}\right) + \frac{i}{2k}\Delta_{\mathbf{x}}\right]\check{b}^\varepsilon, \tag{7}$$

$$\frac{\partial \check{b}^\varepsilon}{\partial z} = -e^{\frac{2ikz}{\varepsilon^2}}\left[\frac{ik}{2\varepsilon}v\left(\frac{z}{\varepsilon^2},\mathbf{x}\right) + \frac{i}{2k}\Delta_{\mathbf{x}}\right]\check{a}^\varepsilon - \left[\frac{ik}{2\varepsilon}v\left(\frac{z}{\varepsilon^2},\mathbf{x}\right) + \frac{i}{2k}\Delta_{\mathbf{x}}\right]\check{b}^\varepsilon. \tag{8}$$

This system is valid in $z \in (0, L)$ and it is complemented with the following boundary conditions at $z = L$ and $z = 0$:

$$\check{b}^\varepsilon(k,z=L,\mathbf{x}) = e^{ikz_0/\varepsilon^2}\check{b}_{\mathrm{inc}}(k,\mathbf{x}), \quad \check{a}^\varepsilon(k,z=0,\mathbf{x}) = R_0\check{b}^\varepsilon(k,z=0,\mathbf{x}), \tag{9}$$

where $R_0 = (Z_0 - 1)/(Z_0 + 1)$ is the reflection coefficient for the interface at $z = 0$ and $Z_0 = \sqrt{K_0\rho_0}$ is the impedance of the left homogeneous half-space. These boundary conditions are obtained from the continuity relations of the fields p^ε and $\mathbf{e}_z \cdot \mathbf{u}^\varepsilon$ at $z = L$ and $z = 0$, which also provide the expressions for the complex amplitudes of the transmitted field $\hat{b}_0^\varepsilon$ in the region $z < 0$ and for the reflected field in the region $z > L$:

$$\check{b}_0^\varepsilon(k,z=0,\mathbf{x}) = T_0\check{b}^\varepsilon(k,z=0,\mathbf{x}), \quad \check{a}_1^\varepsilon(k,z=L,\mathbf{x}) = \check{a}^\varepsilon(k,z=L,\mathbf{x}), \tag{10}$$

where $T_0 = 2Z_0^{1/2}/(1 + Z_0)$ is the transmission coefficient for the interface at $z = 0$. If there is no impedance contrast $Z_0 = 1$, then $T_0 = 1$ and $R_0 = 0$ and the second boundary condition in (9) reads $\check{a}^\varepsilon(k, z = 0, \mathbf{x}) = 0$. This is the radiation condition expressing the fact that there is no wave incoming from $-\infty$. If there is a large impedance contrast $Z_0 \gg 1$ or $Z_0 \ll 1$, then $T_0 \ll 1$ and the reflection coefficient is close to one in absolute value. In particular, if $Z_0 \ll 1$, then $R_0 \simeq -1$ and the second boundary condition in (9) is equivalent to the Dirichlet (reflecting) boundary condition $p^\varepsilon(t, z = 0, \mathbf{x}) = 0$.

We now make use of an invariant imbedding step and introduce the transmission and reflection operators. First we define the lateral Fourier modes

$$\hat{a}^\varepsilon(k,z,\boldsymbol{\kappa}) = \int \check{a}^\varepsilon(k,z,\mathbf{x})e^{-i\boldsymbol{\kappa}\cdot\mathbf{x}}\,d\mathbf{x}, \quad \hat{b}^\varepsilon(k,z,\boldsymbol{\kappa}) = \int \check{b}^\varepsilon(k,z,\mathbf{x})e^{-i\boldsymbol{\kappa}\cdot\mathbf{x}}\,d\mathbf{x}, \tag{11}$$

and make the ansatz

$$\hat{a}^\varepsilon(k,z,\boldsymbol{\kappa}) = \int \widehat{\mathcal{R}}^\varepsilon(k,z,\boldsymbol{\kappa},\boldsymbol{\kappa}')\hat{b}^\varepsilon(k,z,\boldsymbol{\kappa}')\,d\boldsymbol{\kappa}', \quad \hat{b}_0^\varepsilon(k,\boldsymbol{\kappa}) = \int \widehat{\mathcal{T}}^\varepsilon(k,z,\boldsymbol{\kappa},\boldsymbol{\kappa}')\hat{b}^\varepsilon(k,z,\boldsymbol{\kappa}')\,d\boldsymbol{\kappa}'. \tag{12}$$

Using the mode coupling equations (7) and (8) we find

$$\begin{aligned}\frac{d}{dz}\widehat{\mathcal{R}}^\varepsilon(k,z,\boldsymbol{\kappa},\boldsymbol{\kappa}') = {} & e^{-\frac{2ikz}{\varepsilon^2}}\widehat{\mathcal{L}}^\varepsilon(k,z,\boldsymbol{\kappa},\boldsymbol{\kappa}') + e^{\frac{2ikz}{\varepsilon^2}}\int\int \widehat{\mathcal{R}}^\varepsilon(k,z,\boldsymbol{\kappa},\boldsymbol{\kappa}_1)\widehat{\mathcal{L}}^\varepsilon(k,z,\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2)\widehat{\mathcal{R}}^\varepsilon(k,z,\boldsymbol{\kappa}_2,\boldsymbol{\kappa}')\,d\boldsymbol{\kappa}_1\,d\boldsymbol{\kappa}_2 \\ & + \int \widehat{\mathcal{L}}^\varepsilon(k,z,\boldsymbol{\kappa},\boldsymbol{\kappa}_1)\widehat{\mathcal{R}}^\varepsilon(k,z,\boldsymbol{\kappa}_1,\boldsymbol{\kappa}') + \widehat{\mathcal{R}}^\varepsilon(k,z,\boldsymbol{\kappa},\boldsymbol{\kappa}_1)\widehat{\mathcal{L}}^\varepsilon(k,z,\boldsymbol{\kappa}_1,\boldsymbol{\kappa}')\,d\boldsymbol{\kappa}_1,\end{aligned} \tag{13}$$

$$\begin{aligned}\frac{d}{dz}\widehat{\mathcal{T}}^\varepsilon(k,z,\boldsymbol{\kappa},\boldsymbol{\kappa}') = {} & \int \widehat{\mathcal{T}}^\varepsilon(k,z,\boldsymbol{\kappa},\boldsymbol{\kappa}_1)\widehat{\mathcal{L}}^\varepsilon(k,z,\boldsymbol{\kappa}_1,\boldsymbol{\kappa}')\,d\boldsymbol{\kappa}_1 \\ & + e^{\frac{2ikz}{\varepsilon^2}}\int\int \widehat{\mathcal{T}}^\varepsilon(k,z,\boldsymbol{\kappa},\boldsymbol{\kappa}_1)\widehat{\mathcal{L}}^\varepsilon(k,z,\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2)\widehat{\mathcal{R}}^\varepsilon(k,z,\boldsymbol{\kappa}_2,\boldsymbol{\kappa}')\,d\boldsymbol{\kappa}_1\,d\boldsymbol{\kappa}_2,\end{aligned} \tag{14}$$

where we have defined

$$\widehat{\mathcal{L}}^\varepsilon(k,z,\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2) = -\frac{i}{2k}|\boldsymbol{\kappa}_1|^2\delta(\boldsymbol{\kappa}_1-\boldsymbol{\kappa}_2) + \frac{ik}{\varepsilon^2(2\pi)^d}\hat{v}\left(\frac{z}{\varepsilon^2},\boldsymbol{\kappa}_1-\boldsymbol{\kappa}_2\right), \tag{15}$$

with $\hat{v}(z,\boldsymbol{\kappa})$ the partial Fourier transform of $v(z,\mathbf{x})$ defined as in (6). This system is complemented with the initial conditions at $z = 0$:

$$\widehat{\mathcal{R}}^\varepsilon(k,z=0,\boldsymbol{\kappa},\boldsymbol{\kappa}') = R_0\delta(\boldsymbol{\kappa}-\boldsymbol{\kappa}'), \quad \widehat{\mathcal{T}}^\varepsilon(k,z=0,\boldsymbol{\kappa},\boldsymbol{\kappa}') = T_0\delta(\boldsymbol{\kappa}-\boldsymbol{\kappa}').$$

The transmission and reflection operators evaluated at $z = L$ carry all the relevant information about the random medium from the point of view of the transmitted and reflected waves, which are our main quantities of interest.

Our objective in the next sections is to characterize the transmitted wave field

$$p^\varepsilon_{\rm tr}(s,\mathbf{x}) = p^\varepsilon(z_0+\varepsilon^2 s, z=0^-,\mathbf{x}) = \frac{1}{2\pi}\int\int \hat{b}^\varepsilon_0(k,0,\boldsymbol{\kappa}')\,d\boldsymbol{\kappa}' e^{i(\boldsymbol{\kappa}\cdot\mathbf{x}-kz_0/\varepsilon^2-ks)}\,d\boldsymbol{\kappa}\,dk$$
$$= \frac{1}{(2\pi)^{d+1}}\int\int\int \widehat{\mathcal{T}}^\varepsilon(k,L,\boldsymbol{\kappa},\boldsymbol{\kappa}')\hat{b}_{\rm inc}(k,\boldsymbol{\kappa}')\,d\boldsymbol{\kappa}' e^{i(\boldsymbol{\kappa}\cdot\mathbf{x}-ks)}\,d\boldsymbol{\kappa}\,dk, \tag{16}$$

and the reflected wave field

$$p^\varepsilon_{\rm ref}(s,\mathbf{x}) = p^\varepsilon(z_0+L+\varepsilon^2 s, z=L^+,\mathbf{x}) = \frac{1}{2\pi}\int\int \hat{a}^\varepsilon_1(k,L,\boldsymbol{\kappa}')d\boldsymbol{\kappa}' e^{i(\boldsymbol{\kappa}\cdot\mathbf{x}-k(z_0+L)/\varepsilon^2-ks)}\,d\boldsymbol{\kappa}\,dk$$
$$= \frac{1}{(2\pi)^{d+1}}\int\int\int \widehat{\mathcal{R}}^\varepsilon(k,L,\boldsymbol{\kappa},\boldsymbol{\kappa}')\hat{b}_{\rm inc}(k,\boldsymbol{\kappa}')\,d\boldsymbol{\kappa}' e^{i(\boldsymbol{\kappa}\cdot\mathbf{x}-ks)}\,d\boldsymbol{\kappa}\,dk. \tag{17}$$

Note that these wave fields are observed on the time scale of the source and around their respective expected arrival times (z_0 for the transmitted wave, and $z_0 + L$ for the reflected wave, which corresponds to the sum of the travel time from the source z_0 to the interface 0 and the travel time from 0 to L).

3. Random Schrödinger model for the transmission operator

We consider the transmitted field $p^\varepsilon_{\rm tr}$ defined by (16) and use diffusion approximation theorems to identify a random Schrödinger model. The main result is the following.

Proposition 1. *The processes $p^\varepsilon_{tr}(s,\mathbf{x})$ converge in distribution in the space $C^0(\mathbb{R},L^2_w(\mathbb{R}^d))\cap L^2_w(\mathbb{R},L^2_w(\mathbb{R}^d))$ to the limit process*

$$p_{\rm tr}(s,\mathbf{x}) = \frac{1}{2\pi}\int\int \check{\mathcal{T}}(k,L,\mathbf{x},\mathbf{x}')\check{b}_{\rm inc}(k,\mathbf{x}')\,d\mathbf{x}' e^{-iks}\,dk. \tag{18}$$

Here $C^0(\mathbb{R},L^2_w(\mathbb{R}^d))$ is the space of continuous functions in s with values in $L^2_w(\mathbb{R}^d)$ equipped with the weak topology and $L^2_w(\mathbb{R},L^2_w(\mathbb{R}^d)) = L^2_w(\mathbb{R}\times\mathbb{R}^d)$. The operators $\check{\mathcal{T}}(k,z,\mathbf{x},\mathbf{x}')$ are the solutions of the following Itô–Schrödinger diffusion model:

$$d\check{\mathcal{T}}(k,z,\mathbf{x},\mathbf{x}') = \frac{i}{2k}\Delta_{\mathbf{x}'}\check{\mathcal{T}}(k,z,\mathbf{x},\mathbf{x}')dz - \frac{k^2C_{2k}(\mathbf{0})}{8}\check{\mathcal{T}}(k,z,\mathbf{x},\mathbf{x}')dz + \frac{ik}{2}\check{\mathcal{T}}(k,z,\mathbf{x},\mathbf{x}')\circ dB(z,\mathbf{x}'), \tag{19}$$

starting from $\check{\mathcal{T}}(k,0,\mathbf{x},\mathbf{x}') = T_0\delta(\mathbf{x}-\mathbf{x}')$. Here $\circ$ stands for the Stratonovich stochastic integral, $B(z,\mathbf{x})$ is a Brownian field with covariance

$$\mathbb{E}[B(z_1,\mathbf{x}_1)B(z_2,\mathbf{x}_2)] = \min\{z_1,z_2\}\int_{-\infty}^{\infty} C(s,\mathbf{x}_1-\mathbf{x}_2)\,ds, \tag{20}$$

and we have defined

$$C(z,\mathbf{x}) = \mathbb{E}[\nu(z'+z,\mathbf{x}'+\mathbf{x})\nu(z',\mathbf{x}')], \tag{21}$$
$$C_k(\mathbf{x}) = \int_{-\infty}^{\infty} C(z,\mathbf{x})e^{-ikz}\,dz. \tag{22}$$

The moments of the finite-dimensional distributions also converge:

$$\mathbb{E}[p^\varepsilon_{\rm tr}(s_1,\mathbf{x}_1)^{m_1}\cdots p^\varepsilon_{\rm tr}(s_q,\mathbf{x}_q)^{m_q}] \overset{\varepsilon\to 0}{\to} \mathbb{E}[p_{\rm tr}(s_1,\mathbf{x}_1)^{m_1}\cdots p_{\rm tr}(s_q,\mathbf{x}_q)^{m_q}], \tag{23}$$

for any $q\in\mathbb{N}$, $s_1,\dots,s_q\in\mathbb{R}$, $\mathbf{x}_1,\dots,\mathbf{x}_q\in\mathbb{R}^d$, and $m_1,\dots,m_q\in\mathbb{N}$.

The existence and uniqueness problem for the solution of the Itô–Schrödinger model (19) was addressed in [4]. The end of the section is devoted to the proof of this proposition. We shall use a technique similar to the one presented in [9] in the case of randomly layered media. The idea is to study the convergence of a family of moments of the operator $\widehat{\mathcal{T}}^\varepsilon$ that determines the distribution of the transmitted field. The operator $\widehat{\mathcal{T}}^\varepsilon$ itself does not converge to $\widehat{\mathcal{T}}$ (the Fourier transform of $\check{\mathcal{T}}$), but some specific moments of $\widehat{\mathcal{T}}^\varepsilon$ (expectations of products of components with distinct frequencies k) converge to those of $\widehat{\mathcal{T}}$, and this is what is needed.

Step 1. A priori estimates. From (7) and (8) we can check that, for any k, the integral $\int|\check{a}^\varepsilon(k,z,\mathbf{x})|^2 - |\check{b}^\varepsilon(k,z,\mathbf{x})|^2\,d\mathbf{x}$ is conserved in z. Applying this conservation relation at $z=0$ and $z=L$, and taking into account the boundary conditions (9), we obtain

$$\int|\check{a}^\varepsilon(k,L,\mathbf{x})|^2d\mathbf{x} + (1-R_0^2)\int|\check{b}^\varepsilon(k,0,\mathbf{x})|^2d\mathbf{x} = \int|\check{b}_{\rm inc}(k,\mathbf{x})|^2\,d\mathbf{x}.$$

Using now (10) and the identity $R_0^2 + T_0^2 = 1$ we get

$$\int |\check{a}_1^\varepsilon(k, L, \mathbf{x})|^2 \, d\mathbf{x} + \int |\check{b}_0^\varepsilon(k, 0, \mathbf{x})|^2 \, d\mathbf{x} = \int |\check{b}_{\rm inc}(k, \mathbf{x})|^2 \, d\mathbf{x}, \tag{24}$$

which expresses the fact that the power of the incoming wave is fully recovered by the transmitted and reflected waves. Integrating in k and using Parseval's equality gives the total energy conservation relation

$$\int\int |p_{\rm ref}^\varepsilon(s, \mathbf{x})|^2 \, d\mathbf{x}\, ds + \int\int |p_{\rm tr}^\varepsilon(s, \mathbf{x})|^2 \, d\mathbf{x}\, ds = \frac{1}{2\pi}\int\int |\check{b}_{\rm inc}(k, \mathbf{x})|^2 \, d\mathbf{x}\, dk. \tag{25}$$

We first state a priori estimates for our quantities of interest.

Lemma 1. *There exists $C > 0$ such that, uniformly in ε and in s_0, s_1,*

$$\int |p_{\rm tr}^\varepsilon(s_0, \mathbf{x})|^2 \, d\mathbf{x} \leqslant C \text{ and } \int |p_{\rm tr}^\varepsilon(s_1, \mathbf{x}) - p_{\rm tr}^\varepsilon(s_0, \mathbf{x})|^2 \, d\mathbf{x} \leqslant C|s_1 - s_0|. \tag{26}$$

Proof. Using the Sobolev's embedding $L^\infty(\mathbb{R}) \subset H^1(\mathbb{R})$, there exists a constant $C_{\rm sob}$ such that, for any $\mathbf{x}$, $\sup_s |p_{\rm tr}^\varepsilon(s, \mathbf{x})|^2 \leqslant C_{\rm sob}\|p_{\rm tr}^\varepsilon(\cdot, \mathbf{x})\|^2_{H^1(\mathbb{R})}$. Then, using Parseval's equality, we obtain

$$\sup_s |p_{\rm tr}^\varepsilon(s, \mathbf{x})|^2 \leqslant \frac{C_{\rm sob}}{2\pi}\int (1 + k^2)|\check{b}^\varepsilon(k, 0, \mathbf{x})|^2 \, dk.$$

Integrating in $\mathbf{x}$ and using the conservation equation (24) yields the first result of the Lemma:

$$\sup_s \int |p_{\rm tr}^\varepsilon(s, \mathbf{x})|^2 d\mathbf{x} \leqslant \int \sup_s |p_{\rm tr}^\varepsilon(s, \mathbf{x})|^2 \, d\mathbf{x} \leqslant \frac{C_{\rm sob}}{2\pi}\int\int (1 + k^2)|\check{b}_{\rm inc}(k, \mathbf{x})|^2 \, d\mathbf{x}\, dk.$$

By Cauchy–Schwarz inequality, we have

$$|p_{\rm tr}^\varepsilon(s_1, \mathbf{x}) - p_{\rm tr}^\varepsilon(s_0, \mathbf{x})|^2 = \left|\int_{s_0}^{s_1} \frac{\partial p_{\rm tr}^\varepsilon}{\partial s}(s, \mathbf{x})\, ds\right|^2 \leqslant \int_{s_0}^{s_1} ds \int_{s_0}^{s_1} \left|\frac{\partial p_{\rm tr}^\varepsilon}{\partial s}(s, \mathbf{x})\right|^2 ds \leqslant |s_1 - s_0| \int \left|\frac{\partial p_{\rm tr}^\varepsilon}{\partial s}(s, \mathbf{x})\right|^2 ds.$$

The integral in $\mathbf{x}$ of the last term of the inequality can be bounded uniformly as above, which completes the proof. We remark that, in fact, Sobolev's embedding gives that $C^{1/2}(\mathbb{R}) \subset H^1(\mathbb{R})$ providing a result on smoothness in time. □

Step 2. The moments of the finite-dimensional distribution of $p_{tr}^\varepsilon(s, \mathbf{x})$ converge to those of $p_{tr}(s, \mathbf{x})$. Let us fix times $s_1, \ldots, s_q$, positions $\mathbf{x}_1, \ldots, \mathbf{x}_q$, and integers $m_1, \ldots, m_q$. The general moment (23) of $p_{\rm tr}^\varepsilon(s, \mathbf{x})$ can be expressed as the multiple integral

$$\mathbb{E}[p_{\rm tr}^\varepsilon(s_1, \mathbf{x}_1)^{m_1} \cdots p_{\rm tr}^\varepsilon(s_q, \mathbf{x}_q)^{m_q}] = \frac{1}{(2\pi)^{N(d+1)}}\int \cdots \int \prod_{h=1}^q \prod_{j=1}^{m_h} d\boldsymbol{\kappa}'_{h,j} d\boldsymbol{\kappa}_{h,j} dk_{h,j} \prod_{h,j}\left(\hat{b}_{\rm inc}(k_{h,j}, \boldsymbol{\kappa}'_{h,j}) e^{i(\boldsymbol{\kappa}_{h,j}\cdot\mathbf{x}_h - k_{h,j}s_h)}\right)\mathbb{E}\left[\prod_{h,j}\widehat{\mathcal{T}}^\varepsilon(k_{h,j}, L, \boldsymbol{\kappa}_{h,j}, \boldsymbol{\kappa}'_{h,j})\right],$$

for $N = \sum_{h=1}^q m_h$. Therefore, the convergence of the general moment of the transmitted wave field will follow from the convergence of the following specific moments of the transmission operator

$$\mathbb{E}\left[\prod_{j=1}^N \widehat{\mathcal{T}}^\varepsilon(k_j, L, \boldsymbol{\kappa}_j, \boldsymbol{\kappa}'_j)\right]. \tag{27}$$

We call these moments "specific" because we restrict our attention to the case in which the frequencies k_j are all distinct. In Appendix A we use diffusion approximation theorems to deduce that we have

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\prod_j \widehat{\mathcal{T}}^\varepsilon(k_j, L, \boldsymbol{\kappa}_j, \boldsymbol{\kappa}'_j)\right] = \mathbb{E}\left[\prod_j \widehat{\mathcal{T}}(k_j, L, \boldsymbol{\kappa}_j, \boldsymbol{\kappa}'_j)\right],$$

when the right-hand side expectation is taken with respect to the following Itô–Schrödinger model for the transmission operator:

$$d\widehat{\mathcal{T}}(k, z, \boldsymbol{\kappa}, \boldsymbol{\kappa}') = -\frac{k^2(C_0(\mathbf{0}) + C_{2k}(\mathbf{0}))}{8}\widehat{\mathcal{T}}(k, z, \boldsymbol{\kappa}, \boldsymbol{\kappa}')dz - \frac{i|\boldsymbol{\kappa}'|^2}{2k}\widehat{\mathcal{T}}(k, z, \boldsymbol{\kappa}, \boldsymbol{\kappa}')dz + \frac{ik}{2(2\pi)^d}\int \widehat{\mathcal{T}}(k, z, \boldsymbol{\kappa}, \boldsymbol{\kappa}_1) d\widehat{B}(z, \boldsymbol{\kappa}_1 - \boldsymbol{\kappa}')\, d\boldsymbol{\kappa}_1, \tag{28}$$

starting from $\widehat{\mathcal{T}}(k, 0, \boldsymbol{\kappa}, \boldsymbol{\kappa}') = T_0\delta(\boldsymbol{\kappa} - \boldsymbol{\kappa}')$. Here we have used the notations (21) and (22) and the Brownian field $\widehat{B}$ is the partial Fourier transform of the field B so that it has the following operator-valued spatial covariance

$$\mathbb{E}[\widehat{B}(z_1, \boldsymbol{\kappa}_1)\widehat{B}(z_2, \boldsymbol{\kappa}_2)] = \min\{z_1, z_2\}(2\pi)^d \widehat{C}_0(\boldsymbol{\kappa}_1)\delta(\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_2), \tag{29}$$

$$\widehat{C}_k(\boldsymbol{\kappa}) = \int_{-\infty}^{\infty}\int_{\mathbb{R}^d} C(z, \mathbf{x}) e^{-ikz - i\boldsymbol{\kappa}\cdot\mathbf{x}}\, d\mathbf{x}\, dz. \tag{30}$$

Consider next the transmission operator in (28) in the original spatial variables:

$$\check{\mathcal{T}}(k,z,\mathbf{x},\mathbf{x}') = \frac{1}{(2\pi)^d}\int e^{i(\boldsymbol{\kappa}\cdot\mathbf{x}-\boldsymbol{\kappa}'\cdot\mathbf{x}')}\widehat{\mathcal{T}}(k,z,\boldsymbol{\kappa},\boldsymbol{\kappa}')\,d\boldsymbol{\kappa}\,d\boldsymbol{\kappa}'. \tag{31}$$

Then we find that this operator is weakly characterized by the Itô–Schrödinger diffusion model (19). This proves therefore the last statement of the Proposition (the convergence of the moments).

Step 3. Convergence of p_{tr}^{ε} to p_{tr} in $C^0(\mathbb{R}, L_w^2(\mathbb{R}^d)) \cap L_w^2(\mathbb{R}, L_w^2(\mathbb{R}^d))$. Lemma 1 shows that the process $p_{\mathrm{tr}}^{\varepsilon}$ is tight in $C^0(\mathbb{R}, L_w^2(\mathbb{R}^d))$. Moreover, the first estimate also shows that, for any function ϕ in $L^2(\mathbb{R}^d)$ the random processes

$$X_{\phi}^{\varepsilon}(s) = \int p_{\mathrm{tr}}^{\varepsilon}(s,\mathbf{x})\phi(\mathbf{x})\,d\mathbf{x}$$

are uniformly bounded. Therefore, the finite-dimensional distributions are characterized by the moments of the form

$$\mathbb{E}\left[X_{\phi_1}^{\varepsilon}(s_1)^{m_1}\cdots X_{\phi_q}^{\varepsilon}(s_q)^{m_q}\right],$$

where $q\in\mathbb{N}$, $m_1,\ldots,m_q\in\mathbb{N}$, $\phi_1,\ldots,\phi_q\in L^2(\mathbb{R}^d)$. These moments can be written as multiple integrals

$$\mathbb{E}\left[X_{\phi_1}^{\varepsilon}(s_1)^{m_1}\cdots X_{\phi_q}^{\varepsilon}(s_q)^{m_q}\right] = \frac{1}{(2\pi)^{N(d+1)}}\int\cdots\int\prod_{h,j} d\boldsymbol{\kappa}_{h,j}\,d\boldsymbol{\kappa}'_{h,j}\,dk_{h,j}$$
$$\times\prod_{h,j}\left(\hat{b}_{\mathrm{inc}}(k_{h,j},\boldsymbol{\kappa}'_{h,j})\overline{\hat{\phi}_h}(\boldsymbol{\kappa}_{h,j})e^{-ik_{h,j}s_h}\right)\mathbb{E}\left[\prod_{h,j}\check{\mathcal{T}}^{\varepsilon}(k_{h,j},L,\boldsymbol{\kappa}_{h,j},\boldsymbol{\kappa}'_{h,j})\right],$$

for $N=\sum_{h=1}^{q} m_h$, where only the specific moments of the form (27) appear (i.e., moments of products of the transmission operator at distinct k). The convergence of these specific moments therefore implies the convergence of the finite-dimensional distributions, hence the weak convergence in $C^0(\mathbb{R}, L_w^2(\mathbb{R}^d))$. Furthermore, the estimate (25) shows that the processes are tight in $L_w^2(\mathbb{R}, L_w^2(\mathbb{R}^d))$. This proves the first statement of the proposition and completes its proof. □

4. Generalized Schrödinger model for the reflection operator

In this section we shift our attention to the reflected wave $p_{\mathrm{ref}}^{\varepsilon}$ defined by (17). We again use diffusion approximation theorems to identify a coupled random Schrödinger model. By an argument as presented in Section 3 regarding the transmitted field we find the following result.

Proposition 2. *The processes $p_{\mathrm{ref}}^{\varepsilon}(s,\mathbf{x})$ converge in distribution in the space $C^0(\mathbb{R}, L_w^2(\mathbb{R}^d)) \cap L_w^2(\mathbb{R}, L_w^2(\mathbb{R}^d))$ to the limit process*

$$p_{\mathrm{ref}}(s,\mathbf{x}) = \frac{1}{2\pi}\int\int\check{\mathcal{R}}(k,L,\mathbf{x},\mathbf{x}')\check{b}_{\mathrm{inc}}(k,\mathbf{x}')\,d\mathbf{x}'e^{-iks}\,dk. \tag{32}$$

The operators $\check{\mathcal{R}}(k,z,\mathbf{x},\mathbf{x}')$ are the solutions of the following Itô–Schrödinger diffusion model

$$d\check{\mathcal{R}}(k,z,\mathbf{x},\mathbf{x}') = \frac{i}{2k}(\Delta_{\mathbf{x}}+\Delta_{\mathbf{x}'})\check{\mathcal{R}}(k,z,\mathbf{x},\mathbf{x}')dz - \frac{k^2C_{2k}(\mathbf{0})}{4}\check{\mathcal{R}}(k,z,\mathbf{x},\mathbf{x}')dz + \frac{ik}{2}\check{\mathcal{R}}(k,z,\mathbf{x},\mathbf{x}')\circ(dB(z,\mathbf{x})+dB(z,\mathbf{x}')), \tag{33}$$

starting from $\check{\mathcal{R}}(k,0,\mathbf{x},\mathbf{x}') = R_0\delta(\mathbf{x}-\mathbf{x}')$. The moments of the finite-dimensional distributions also converge

$$\mathbb{E}[p_{\mathrm{ref}}^{\varepsilon}(s_1,\mathbf{x}_1)^{m_1}\cdots p_{\mathrm{ref}}^{\varepsilon}(s_q,\mathbf{x}_q)^{m_q}] \overset{\varepsilon\to 0}{\to} \mathbb{E}[p_{\mathrm{ref}}(s_1,\mathbf{x}_1)^{m_1}\cdots p_{\mathrm{ref}}(s_q,\mathbf{x}_q)^{m_q}], \tag{34}$$

for any $q\in\mathbb{N}$, $s_1,\ldots,s_q\in\mathbb{R}$, $\mathbf{x}_1,\ldots,\mathbf{x}_q\in\mathbb{R}^d$, and $m_1,\ldots,m_q\in\mathbb{N}$.

Proof. The proof follows the same line as the one of Proposition 1 for the transmitted field. In particular, the moments (34) are characterized by the specific moments of the reflection operator

$$\mathbb{E}\left[\prod_{j=1}^{N}\widehat{\mathcal{R}}^{\varepsilon}(k_j,L,\boldsymbol{\kappa}_j,\boldsymbol{\kappa}'_j)\right], \tag{35}$$

for different frequencies k_j. In Appendix B we use diffusion approximation theorems to show that indeed

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\prod_j\widehat{\mathcal{R}}^{\varepsilon}(k_j,L,\boldsymbol{\kappa}_j,\boldsymbol{\kappa}'_j)\right] = \mathbb{E}\left[\prod_j\widehat{\mathcal{R}}(k_j,L,\boldsymbol{\kappa}_j,\boldsymbol{\kappa}'_j)\right],$$

when the right-hand side expectation is taken with respect to the following coupled Itô–Schrödinger model for the reflection operator:

$$d\widehat{\mathcal{R}}(k,z,\boldsymbol{\kappa},\boldsymbol{\kappa}') = -\frac{i(|\boldsymbol{\kappa}|^2+|\boldsymbol{\kappa}'|^2)}{2k}\widehat{\mathcal{R}}(k,z,\boldsymbol{\kappa},\boldsymbol{\kappa}')dz - \frac{k^2(C_0(\mathbf{0})+C_{2k}(\mathbf{0}))}{4}\widehat{\mathcal{R}}(k,z,\boldsymbol{\kappa},\boldsymbol{\kappa}')dz \\ -\frac{k^2}{4(2\pi)^d}\int \widehat{C}_0(\boldsymbol{\kappa}_1)\widehat{\mathcal{R}}(k,z,\boldsymbol{\kappa}-\boldsymbol{\kappa}_1,\boldsymbol{\kappa}'-\boldsymbol{\kappa}_1)\,d\boldsymbol{\kappa}_1\,dz \\ +\frac{ik}{2(2\pi)^d}\int\Big(\widehat{\mathcal{R}}(k,z,\boldsymbol{\kappa},\boldsymbol{\kappa}_1)d\widehat{B}(z,\boldsymbol{\kappa}_1-\boldsymbol{\kappa}')+\widehat{\mathcal{R}}(k,z,\boldsymbol{\kappa}_1,\boldsymbol{\kappa}')\,d\widehat{B}(z,\boldsymbol{\kappa}-\boldsymbol{\kappa}_1)\Big)\,d\boldsymbol{\kappa}_1, \quad (36)$$

starting from $\widehat{\mathcal{R}}(k,0,\boldsymbol{\kappa},\boldsymbol{\kappa}') = R_0\delta(\boldsymbol{\kappa}-\boldsymbol{\kappa}')$. Here $\widehat{B}$ is the Brownian field whose distribution is characterized by the covariance (29). We consider next the reflection operator in the original spatial variables:

$$\check{\mathcal{R}}(k,z,\mathbf{x},\mathbf{x}') = \frac{1}{(2\pi)^d}\int e^{i(\boldsymbol{\kappa}\cdot\mathbf{x}-\boldsymbol{\kappa}'\cdot\mathbf{x}')}\widehat{\mathcal{R}}(k,z,\boldsymbol{\kappa},\boldsymbol{\kappa}')\,d\boldsymbol{\kappa}\,d\boldsymbol{\kappa}', \quad (37)$$

and find that this operator is weakly characterized by the diffusion model (33). □

Regarding the above results we remark first that the limit problems for the reflection operator and for the transmission operator are linear. In fact, the nonlinear terms in (13) and (14) are associated with random backscattering. In the asymptotic regime we are considering, and as far as the reflected wave front $p^{\varepsilon}_{\rm ref}$ is concerned, the nonlinear terms in (13) only manifest themselves via the damping term involving $C_{2k}(\mathbf{0})$ in (33). Similarly for the transmitted wave front $p^{\varepsilon}_{\rm tr}$, the nonlinear terms in (14) only manifest themselves via the damping term involving $C_{2k}(\mathbf{0})$ in (19).

We remark second that, in Propositions 1 and 2, we cannot expect a stronger convergence result. This remark is motivated by the observation that the limit processes $p_{\rm tr}$ and $p_{\rm ref}$ defined by (18) and (32) do not satisfy the energy conservation equation (25). Indeed we have

$$\int\int \mathbb{E}[|p_{\rm ref}(s,\mathbf{x})|^2]d\mathbf{x}ds + \int\int \mathbb{E}[|p_{\rm tr}(s,\mathbf{x})|^2]\,d\mathbf{x}\,ds = \frac{1}{2\pi}\int\int |\check{b}_{\rm inc}(k,\mathbf{x})|^2\left(R_0^2 e^{-\frac{C_{2k}(\mathbf{0})k^2L}{2}} + T_0^2 e^{-\frac{C_{2k}(\mathbf{0})k^2L}{4}}\right)d\mathbf{x}\,dk \\ < \frac{1}{2\pi}\int\int |\check{b}_{\rm inc}(k,\mathbf{x})|^2\,d\mathbf{x}\,dk.$$

The damping factor $\exp(-C_{2k}(\mathbf{0})k^2L/2)$ represents the leading-order effect of randomly backscattered energy described by the nonlinear terms in Eq. (13) for the reflection operator. Similarly the damping factor $\exp(-C_{2k}(\mathbf{0})k^2L/4)$ represents the leading-order effect of randomly backscattered energy described by the nonlinear terms in Eq. (14) for the transmission operator. The term $C_{2k}(\mathbf{0})$ in the damping factor is the spectrum of the medium fluctuations evaluated at the wave vector difference for forward and backward traveling waves and represents intensity of backscattering. We remark finally that this backscattered energy spreads out as an incoherent long and small-amplitude coda and will not accumulate again in the wave front that we are analyzing, this is the mechanism that gives damping in our representation.

5. The Wigner distribution for the transmitted wave

In order to identify different propagation regimes, we will write the Wigner distribution in a dimensionless form. First we introduce the dimensionless autocorrelation function $\mathcal{C}$ of the fluctuations of the random medium

$$C(z,\mathbf{x}) = \sigma^2\mathcal{C}\left(\frac{z}{l_z},\frac{\mathbf{x}}{l_x}\right),$$

where σ is the standard deviation of the fluctuations of the random medium, l_z (respectively, l_x) is the longitudinal (respectively, transverse) correlation radius of the medium. In this situation we have

$$C_0(\mathbf{x}) = \sigma^2 l_z\mathcal{C}_0\left(\frac{\mathbf{x}}{l_x}\right), \quad \widehat{C}_0(\mathbf{u}) = \sigma^2 l_z l_x^d\widehat{\mathcal{C}}_0(\mathbf{u}l_x).$$

We assume next that the power spectral density $\widehat{\mathcal{C}}_0(\mathbf{u})$ decays fast enough so that $\int|\mathbf{u}|^2\widehat{\mathcal{C}}_0(\mathbf{u})\,d\mathbf{u}$ is finite. This means that the autocorrelation function $\mathcal{C}_0(\mathbf{x})$ is at least twice differentiable at $\mathbf{x}=\mathbf{0}$, which corresponds to a smooth random medium. For simplicity, we assume also that the random fluctuations are isotropic in the transverse directions, in the sense that the autocorrelation function $\mathcal{C}_0(\mathbf{x})$ depends only on $|\mathbf{x}|$.

We now consider two frequencies k_1 and k_2 in a frequency band centered at k and we define the two-frequency Wigner distribution of the transmission operator by

$$W^T_{k_1,k_2}(z,\mathbf{x},\mathbf{x}',\mathbf{q},\mathbf{q}') = e^{\frac{C_{2k_1}(\mathbf{0})k_1^2+C_{2k_2}(\mathbf{0})k_2^2}{8}z}\int\int e^{-i(\mathbf{q}\cdot\mathbf{y}+\mathbf{q}'\cdot\mathbf{y}')}\mathbb{E}\left[\check{\mathcal{T}}\left(k_1,z,\frac{\sqrt{k}}{\sqrt{k_1}}(\mathbf{x}+\frac{\mathbf{y}}{2}),\frac{\sqrt{k}}{\sqrt{k_1}}\left(\mathbf{x}'+\frac{\mathbf{y}'}{2}\right)\right)\right. \\ \left.\times\overline{\check{\mathcal{T}}}\left(k_2,z,\frac{\sqrt{k}}{\sqrt{k_2}}(\mathbf{x}-\frac{\mathbf{y}}{2}),\frac{\sqrt{k}}{\sqrt{k_2}}\left(\mathbf{x}'-\frac{\mathbf{y}'}{2}\right)\right)\right]d\mathbf{y}\,d\mathbf{y}'. \quad (38)$$

Note that we have taken out the damping terms proportional to $C_{2k_j}(\mathbf{0})$ in this definition. Using the stochastic equation (19) and Itô's formula, we find that the Wigner distribution satisfies the closed system

$$\frac{\partial W^T_{k_1,k_2}}{\partial z} + \frac{\mathbf{q}'}{k} \cdot \nabla_{\mathbf{x}'} W^T_{k_1,k_2} = -\frac{C_0(\mathbf{0})(k_1^2+k_2^2)}{8} \mathcal{W}^T_{k_1,k_2} + \frac{k_1 k_2}{4(2\pi)^d} \int \widehat{C}_0(\mathbf{u}) W^T_{k_1,k_2}\left(z, \mathbf{x}, \mathbf{x}', \mathbf{q}, \mathbf{q}' - \frac{1}{2}\left(\frac{\sqrt{k}}{\sqrt{k_1}} + \frac{\sqrt{k}}{\sqrt{k_2}}\right)\mathbf{u}\right) e^{i\mathbf{u}\cdot\mathbf{x}'\left(\frac{\sqrt{k}}{\sqrt{k_1}} - \frac{\sqrt{k}}{\sqrt{k_2}}\right)} d\mathbf{u}, \tag{39}$$

starting from $W^T_{k_1,k_2}(z=0,\mathbf{x},\mathbf{x}',\mathbf{q},\mathbf{q}') = T_0^2(4\pi^2 k_1 k_2/k^2)^{d/2}\delta(\mathbf{x}-\mathbf{x}')\delta(\mathbf{q}+\mathbf{q}')$. It is possible to solve this system and to find an integral representation for the two-frequency Wigner distribution using the approach of [5]. However, we aim at focusing on spatial aspects in the next sections, and we shall simplify the algebra by assuming that the bandwidth B of the incoming wave is small. To describe this regime it is convenient to introduce

$$\beta = \frac{\sigma^2 k_0^2 L l_z}{4}, \quad \alpha = \frac{L}{k_0 l_x^2}, \quad \alpha_0 = \frac{L}{k_0 r_0^2}, \tag{40}$$

where k_0 is the carrier wavenumber, r_0 is the initial beam width and where β describes the intensity of forward scattering, while α and α_0 represent the intensities of lateral scattering on, respectively, the scales of the medium variations and the input beam. We assume that the bandwidth B of the incoming wave (with carrier wavenumber k_0) is small in the sense that

$$B \ll B_c, \quad B_c := k_0 \min(1, \alpha^{-1}, \alpha_0^{-1}, \beta^{-1}). \tag{41}$$

If k_1, k_2 lie in the spectrum of the incoming wave, we then find that the two-frequency Wigner distribution $W^T_{k_1,k_2}$ can be approximated by the simplified Wigner distribution W^T that depends only on the carrier wavenumber k_0 and not on the lag $k_1 - k_2$ and that satisfies

$$\frac{\partial W^T}{\partial z} + \frac{\mathbf{q}'}{k_0} \cdot \nabla_{\mathbf{x}'} W^T = \frac{k_0^2}{4(2\pi)^d} \int \widehat{C}_0(\mathbf{u}) \left[W^T(z,\mathbf{x},\mathbf{x}',\mathbf{q},\mathbf{q}'-\mathbf{u}) - W^T(z,\mathbf{x},\mathbf{x}',\mathbf{q},\mathbf{q}')\right] d\mathbf{u}, \tag{42}$$

starting from $W^T(z=0,\mathbf{x},\mathbf{x}',\mathbf{q},\mathbf{q}') = (2\pi)^d T_0^2 \delta(\mathbf{x}-\mathbf{x}')\delta(\mathbf{q}+\mathbf{q}')$. By taking a Fourier transform in $\mathbf{q}'$ and $\mathbf{x}'$, we obtain a transport equation that can be integrated and we find the following integral representation for W^T:

$$W^T(z,\mathbf{x},\mathbf{x}',\mathbf{q},\mathbf{q}') = \frac{T_0^2}{(2\pi)^d} \int\int e^{-i(\mathbf{q}'+\mathbf{q})\cdot\mathbf{a} - i(\mathbf{x}'-\mathbf{x}+\frac{\mathbf{q}}{k_0}z)\cdot\mathbf{b}} e^{\frac{k_0^2}{4}\int_0^z C_0(\mathbf{a}+\frac{\mathbf{b}}{k_0}z') - C_0(\mathbf{0}) dz'} d\mathbf{a}\, d\mathbf{b}. \tag{43}$$

5.1. Slow transverse variations

We want to analyze the regime in which the transverse correlation length l_x of the medium is larger than the beam width r_0. We introduce normalized coordinates and write the Wigner distribution in the form

$$W^T(z,\mathbf{x},\mathbf{x}',\mathbf{q},\mathbf{q}') = T_0^2(2\pi)^d \mathcal{W}^T\left(\frac{z}{L}, \frac{\mathbf{x}}{r_0}, \frac{\mathbf{x}'}{r_0}, \mathbf{q}r_0, \mathbf{q}'r_0\right),$$

where $\mathcal{W}^T$ satisfies

$$\frac{\partial \mathcal{W}^T}{\partial \zeta} + \frac{\alpha l_x^2}{r_0^2}\mathbf{q}' \cdot \nabla_{\mathbf{x}'} \mathcal{W}^T = \frac{\beta}{(2\pi)^d} \int \widehat{C}_0(\mathbf{u}) \left[\mathcal{W}^T\left(z,\mathbf{x},\mathbf{x}',\mathbf{q},\mathbf{q}' - \frac{r_0}{l_x}\mathbf{u}\right) - \mathcal{W}^T(z,\mathbf{x},\mathbf{x}',\mathbf{q},\mathbf{q}')\right] d\mathbf{u}, \tag{44}$$

starting from $\mathcal{W}^T(\zeta=0,\mathbf{x},\mathbf{x}',\mathbf{q},\mathbf{q}') = \delta(\mathbf{x}-\mathbf{x}')\delta(\mathbf{q}+\mathbf{q}')$. We assume in this section that

(1) $r_0 \ll l_x$, which means that the transverse correlation length of the medium is large,
(2) $k_0 r_0^2 \sim L$, which means that diffractive effects are of order one.

These conditions are equivalent to $\alpha_0 = \alpha l_x^2/r_0^2 \sim 1$ and $\alpha \ll 1$. Therefore, we can expand the integral of the right-hand side of (44) with respect to the small parameter r_0/l_x and we obtain the simplified system

$$\frac{\partial \mathcal{W}^T}{\partial \zeta} + \frac{\alpha l_x^2}{r_0^2}\mathbf{q}' \cdot \nabla_{\mathbf{x}'} \mathcal{W}^T = \frac{\beta r_0^2}{2 l_x^2} \mathcal{D} \Delta_{\mathbf{q}'} \mathcal{W}^T, \tag{45}$$

where

$$\mathcal{D} = \frac{1}{d(2\pi)^d} \int \widehat{C}_0(\mathbf{u}) |\mathbf{u}|^2 d\mathbf{u} = -\frac{1}{d} \Delta_{\mathbf{x}} C_0(\mathbf{0}). \tag{46}$$

Here we have used the fact that $\int \mathbf{u}\widehat{\mathcal{C}}(\mathbf{u})\,d\mathbf{u} = \mathbf{0}$ since $\widehat{\mathcal{C}}(\mathbf{u})$ is an even function. In the original variables, the Wigner distribution W^T solves

$$\frac{\partial W^T}{\partial z} + \frac{\mathbf{q}'}{k_0}\cdot\nabla_{\mathbf{x}'}W^T = \frac{k_0^2 D}{8}\Delta_{\mathbf{q}'}W^T, \tag{47}$$

where

$$D = \frac{1}{d(2\pi)^d}\int \widehat{C}_0(\mathbf{u})|\mathbf{u}|^2\,d\mathbf{u} = -\frac{1}{d}\Delta_{\mathbf{x}}C_0(\mathbf{0}) = \frac{\sigma^2 l_z}{l_x^2}\mathcal{D},$$

and it has the form

$$W^T(z,\mathbf{x},\mathbf{x}',\mathbf{q},\mathbf{q}') = T_0^2\left(\frac{192}{D^2k_0^2z^4}\right)^{d/2} e^{-\frac{2|\mathbf{q}+\mathbf{q}'|^2}{k_0^2 Dz}}\, e^{-\frac{24|\mathbf{x}-\mathbf{x}'-\frac{\mathbf{q}-\mathbf{q}'}{2k_0}z|^2}{Dz^3}}. \tag{48}$$

This solution is obtained by solving the advection-diffusion equation (47) after Fourier transforming in $\mathbf{q}'$ and $\mathbf{x}'$ (which gives a transport equation). It is also possible to compute the limit of the integral representation (43) directly. The Wigner distribution contains all the relevant information needed to understand the main properties of the transmitted wave, as we shall see in Section 7.

5.2. Rapid transverse variations

We want now to analyze the regime in which the transverse correlation length l_x of the medium is smaller than the beam width r_0. More exactly, we assume in this section that

(1) $r_0 \gg l_x$, which means that the transverse correlation length of the medium is small,
(2) $k_0 r_0 l_x \sim L$, which means that diffractive effects are of order one. Note that we have not yet established the fact that diffraction plays a role for a propagation distance L of the order of $k_0 r_0 l_x$ which is smaller than the usual Rayleigh length $k_0 r_0^2$. However, this is a well-known result [7,12], and we shall deduce it here in the analytic framework that we have set forth.

The two conditions $r_0 \gg l_x$ and $k_0 r_0 l_x \sim L$ are equivalent to $\alpha \gg 1$ and $\alpha l_x/r_0 \sim 1$.

The Wigner distribution can now be written in the form

$$W^T(z,\mathbf{x},\mathbf{x}',\mathbf{q},\mathbf{q}') = T_0^2(2\pi)^d\mathcal{W}^T\left(\frac{z}{L},\frac{\mathbf{x}}{r_0},\frac{\mathbf{x}'}{r_0},\mathbf{q}l_x,\mathbf{q}'l_x\right),$$

where $\mathcal{W}^T$ satisfies

$$\frac{\partial \mathcal{W}^T}{\partial \zeta} + \frac{\alpha l_x}{r_0}\mathbf{q}'\cdot\nabla_{\mathbf{x}'}\mathcal{W}^T = \frac{\beta}{(2\pi)^d}\int \widehat{\mathcal{C}}_0(\mathbf{u})\left[\mathcal{W}^T(z,\mathbf{x},\mathbf{x}',\mathbf{q},\mathbf{q}'-\mathbf{u}) - \mathcal{W}^T(z,\mathbf{x},\mathbf{x}',\mathbf{q},\mathbf{q}')\right]d\mathbf{u}, \tag{49}$$

starting from $\mathcal{W}^T(\zeta=0,\mathbf{x},\mathbf{x}',\mathbf{q},\mathbf{q}') = \delta(\mathbf{x}-\mathbf{x}')\delta(\mathbf{q}+\mathbf{q}')$. It is clear from this equation that diffractive effects (characterized by the term $\mathbf{q}'\cdot\nabla_{\mathbf{x}'}$) are of order one in this regime, in which $\alpha l_x r_0 \sim 1$. We can also consider the integral representation (43) and write it in dimensionless form

$$\mathcal{W}^T(\zeta,\mathbf{x},\mathbf{x}',\mathbf{q},\mathbf{q}') = \frac{1}{(4\pi^2\alpha)^d}\int\int e^{-i(\mathbf{q}'+\mathbf{q})\cdot\mathbf{a}-i\left(\frac{r_0}{\alpha l_x}(\mathbf{x}'-\mathbf{x})+\mathbf{q}\zeta\right)\cdot\mathbf{b}}\, e^{\beta\int_0^\zeta \mathcal{C}_0(\mathbf{a}+\mathbf{b}\zeta')-\mathcal{C}_0(0)\,d\zeta'}\,d\mathbf{a}d\mathbf{b}. \tag{50}$$

If additionally, we assume that $\beta \gg 1$, then we obtain an expression which, in the original variables, is exactly (48). Remember that (48) was obtained in the small-α regime. The fact that the large-β behavior is independent of α can be seen directly from (43) and (50). Using the fact that the structure function $\mathbf{a}\mapsto\mathcal{C}_0(\mathbf{0})-\mathcal{C}_0(\mathbf{a})$ is smooth and attains its minimum at the origin, we find that when β is large, the main contribution to the exponential integral is concentrated at small $\mathbf{a}+\mathbf{b}z/k_0$:

$$e^{\frac{k_0^2}{4}\int_0^z C_0(\mathbf{a}+\frac{\mathbf{b}}{k_0}z')-C_0(\mathbf{0})\,dz'} \simeq e^{-\frac{k_0^2 D}{8}\int_0^z\left|\mathbf{a}+\frac{\mathbf{b}}{k_0}z'\right|^2 dz'},$$

and the integration gives (48).

6. The Wigner distribution for the reflected wave

We define the two-frequency Wigner distribution of the reflection operator by

$$W^R_{k_1,k_2}(z,\mathbf{x},\mathbf{x}',\mathbf{q},\mathbf{q}') = e^{\frac{C_{2k_1}(\mathbf{0})k_1^2+C_{2k_2}(\mathbf{0})k_2^2}{4}z}\int\int e^{-i(\mathbf{q}\cdot\mathbf{y}+\mathbf{q}'\cdot\mathbf{y}')}\mathbb{E}\left[\check{\mathcal{R}}\left(k_1,z,\frac{\sqrt{k}}{\sqrt{k_1}}(\mathbf{x}+\frac{\mathbf{y}}{2}),\frac{\sqrt{k}}{\sqrt{k_1}}\left(\mathbf{x}'+\frac{\mathbf{y}'}{2}\right)\right)\right.$$
$$\left.\times\overline{\check{\mathcal{R}}}\left(k_2,z,\frac{\sqrt{k}}{\sqrt{k_2}}(\mathbf{x}-\frac{\mathbf{y}}{2}),\frac{\sqrt{k}}{\sqrt{k_2}}\left(\mathbf{x}'-\frac{\mathbf{y}'}{2}\right)\right)\right]d\mathbf{y}\,d\mathbf{y}'. \tag{51}$$

If the bandwidth of the incoming wave satisfies (41) and if k_1, k_2 lie in the spectrum of the wave, then we find by using (33) that the two-frequency Wigner distribution $W^R_{k_1,k_2}$ can be approximated by the simplified Wigner distribution W^R that depends only on the carrier wavenumber k_0 and not on the lag $k_1 - k_2$ and that satisfies the closed system

$$\frac{\partial W^R}{\partial z}+\frac{\mathbf{q}}{k_0}\cdot\nabla_{\mathbf{x}}W^R+\frac{\mathbf{q}'}{k_0}\cdot\nabla_{\mathbf{x}'}W^R = \frac{k_0^2}{4(2\pi)^d}\int\widehat{C}_0(\mathbf{u})\Big[W^R(z,\mathbf{x},\mathbf{x}',\mathbf{q}-\mathbf{u},\mathbf{q}')+W^R(z,\mathbf{x},\mathbf{x}',\mathbf{q},\mathbf{q}'-\mathbf{u})$$
$$+2W^R\left(z,\mathbf{x},\mathbf{x}',\mathbf{q}-\frac{1}{2}\mathbf{u},\mathbf{q}'-\frac{1}{2}\mathbf{u}\right)\cos\left(\mathbf{u}\cdot(\mathbf{x}-\mathbf{x}')\right)$$
$$-2W^R\left(z,\mathbf{x},\mathbf{x}',\mathbf{q}-\frac{1}{2}\mathbf{u},\mathbf{q}'+\frac{1}{2}\mathbf{u}\right)\cos\left(\mathbf{u}\cdot(\mathbf{x}-\mathbf{x}')\right)-2W^R(z,\mathbf{x},\mathbf{x}',\mathbf{q},\mathbf{q}')\Big]d\mathbf{u},$$

starting from $W^R(z=0,\mathbf{x},\mathbf{x}',\mathbf{q},\mathbf{q}') = R_0^2(2\pi)^d\delta(\mathbf{x}-\mathbf{x}')\delta(\mathbf{q}+\mathbf{q}')$.

6.1. Slow transverse variations

In the regime in which the transverse correlation length of the medium is larger than the beam width ($r_0 \ll l_x$ and $k_0 r_0^2 \sim L$), as in Section 5.1, we obtain the simplified system

$$\frac{\partial W^R}{\partial z}+\frac{\mathbf{q}}{k_0}\cdot\nabla_{\mathbf{x}}W^R+\frac{\mathbf{q}'}{k_0}\cdot\nabla_{\mathbf{x}'}W^R = \frac{k_0^2 D}{8}(\nabla_{\mathbf{q}}+\nabla_{\mathbf{q}'})\cdot(\nabla_{\mathbf{q}}+\nabla_{\mathbf{q}'})W^R. \tag{52}$$

This system can be integrated and the solution reads

$$W^R(z,\mathbf{x},\mathbf{x}',\mathbf{q},\mathbf{q}') = R_0^2\left(\frac{2\pi}{k_0^2 Dz}\right)^{d/2}\delta\left(\mathbf{x}-\mathbf{x}'-\frac{\mathbf{q}-\mathbf{q}'}{k_0}z\right)e^{-\frac{|\mathbf{q}+\mathbf{q}'|^2}{2k_0^2 Dz}}. \tag{53}$$

6.2. Rapid transverse variations

The regime in which $r_0 \gg l_x$ and $k_0 r_0 l_x \sim L$, as in Section 5.2, is more delicate and more interesting to study because W^R exhibits a multi-scale behavior. So we shall first cast the Wigner distribution in a suitable dimensionless form. We consider the following Fourier transform V^R of the Wigner distribution W^R:

$$W^R(z,\mathbf{x},\mathbf{x}',\mathbf{q},\mathbf{q}') = \frac{1}{(2\pi)^d}\int V^R\left(z,\frac{\mathbf{q}+\mathbf{q}'}{2},\mathbf{q}-\mathbf{q}',\mathbf{c}\right)e^{i\mathbf{c}\cdot(\mathbf{x}'-\mathbf{x})}d\mathbf{c},$$

which we introduce because the stationary maps that we will identify in Lemma 2 in the asymptotic regime $\alpha\to\infty$ have simple representations in this new frame. Note also that this ansatz incorporates the fact that W^R does not depend on $\mathbf{x}+\mathbf{x}'$, only on $\mathbf{x}-\mathbf{x}'$. The Fourier-transformed operator $V^R(z,\mathbf{a},\mathbf{b},\mathbf{c})$ has the form

$$V^R(z,\mathbf{a},\mathbf{b},\mathbf{c}) = R_0^2(\pi l_x)^d e^{\frac{iz}{k_0}\mathbf{b}\cdot\mathbf{c}}\mathcal{V}^R\left(\frac{z}{L},\mathbf{a}l_x,\mathbf{b}l_x,\mathbf{c}l_x\right),$$

where $\mathcal{V}^R$ is the solution of the dimensionless system

$$\frac{\partial\mathcal{V}^R}{\partial\zeta} = \frac{\beta}{(2\pi)^d}\int\widehat{C}_0(\mathbf{u})\Big[\mathcal{V}^R\left(\zeta,\mathbf{q}-\frac{1}{2}\mathbf{u},\mathbf{r}-\mathbf{u},\mathbf{s}\right)e^{-i\alpha\mathbf{s}\cdot\mathbf{u}\zeta}+\mathcal{V}^R\left(\zeta,\mathbf{q}-\frac{1}{2}\mathbf{u},\mathbf{r}+\mathbf{u},\mathbf{s}\right)e^{i\alpha\mathbf{s}\cdot\mathbf{u}\zeta}+\mathcal{V}^R\left(\zeta,\mathbf{q}-\frac{1}{2}\mathbf{u},\mathbf{r},\mathbf{s}-\mathbf{u}\right)e^{-i\alpha\mathbf{r}\cdot\mathbf{u}\zeta}$$
$$+\mathcal{V}^R\left(\zeta,\mathbf{q}-\frac{1}{2}\mathbf{u},\mathbf{r},\mathbf{s}+\mathbf{u}\right)e^{i\alpha\mathbf{r}\cdot\mathbf{u}\zeta}-\mathcal{V}^R\left(\zeta,\mathbf{q}-\frac{1}{2}\mathbf{u},\mathbf{r}-\mathbf{u},\mathbf{s}+\mathbf{u}\right)e^{i\alpha[(\mathbf{r}-\mathbf{s})\cdot\mathbf{u}-|\mathbf{u}|^2]\zeta}$$
$$-\mathcal{V}^R\left(\zeta,\mathbf{q}-\frac{1}{2}\mathbf{u},\mathbf{r}-\mathbf{u},\mathbf{s}-\mathbf{u}\right)e^{-i\alpha[(\mathbf{r}+\mathbf{s})\cdot\mathbf{u}+|\mathbf{u}|^2]\zeta}-2\mathcal{V}^R(\zeta,\mathbf{q},\mathbf{r},\mathbf{s})\Big]d\mathbf{u}, \tag{54}$$

starting from $\mathcal{V}^R(\zeta,\mathbf{q},\mathbf{r},\mathbf{s}) = \delta(\mathbf{q})$. The parameters α and β are given by (40). The regime $\alpha \to \infty$ corresponds to rapid transverse fluctuations of the random medium. In (54) this regime gives rise to rapid phases. The following proposition describes the asymptotic behavior of $\mathcal{V}^R$ as $\alpha \to \infty$. The presence of singular layers at $\mathbf{r} = \mathbf{0}$ and at $\mathbf{s} = \mathbf{0}$ requires particular attention.

Lemma 2.

1. *For any* $\mathbf{r}\neq\mathbf{0}$, $\mathbf{s}\neq\mathbf{0}$*:*
$$\mathcal{V}^R(\zeta,\mathbf{q},\mathbf{r},\mathbf{s}) \overset{\alpha\to\infty}{\longrightarrow} \delta(\mathbf{q})e^{-2\beta\mathcal{C}_0(\mathbf{0})\zeta}. \tag{55}$$
2. *For any* $\mathbf{s}\neq\mathbf{0}$ *we have* $\mathcal{V}^R(\zeta,\mathbf{q},\frac{\mathbf{r}}{\alpha},\mathbf{s}) \overset{\alpha\to\infty}{\longrightarrow} \mathcal{V}^R_{\mathbf{r}}(\zeta,\mathbf{q})$ *where* $\mathcal{V}^R_{\mathbf{r}}(\zeta,\mathbf{q})$ *is solution of*
$$\frac{\partial\mathcal{V}^R_{\mathbf{r}}}{\partial\zeta} = \frac{2\beta}{(2\pi)^d}\int \widehat{\mathcal{C}}_0(\mathbf{u})\left[\mathcal{V}^R_{\mathbf{r}}\left(\zeta,\mathbf{q}-\frac{1}{2}\mathbf{u}\right)\cos(\mathbf{r}\cdot\mathbf{u}\zeta) - \mathcal{V}^R_{\mathbf{r}}(\zeta,\mathbf{q})\right]d\mathbf{u}, \tag{56}$$
and is given explicitly by
$$\mathcal{V}^R_{\mathbf{r}}(\zeta,\mathbf{q}) = \frac{1}{(2\pi)^d}\int e^{-i\mathbf{q}\cdot\mathbf{u}}e^{\beta\int_{-\zeta}^{\zeta}\mathcal{C}_0\left(\frac{\mathbf{u}}{2}+\mathbf{r}\zeta'\right)-\mathcal{C}_0(\mathbf{0})d\zeta'}\,d\mathbf{u}. \tag{57}$$
Similarly, for any $\mathbf{r}\neq\mathbf{0}$ *we have* $\mathcal{V}^R(\zeta,\mathbf{q},\mathbf{r},\frac{\mathbf{s}}{\alpha}) \overset{\alpha\to\infty}{\longrightarrow} \mathcal{V}^R_{\mathbf{s}}(\zeta,\mathbf{q})$.
3. *For any* $\mathbf{r}$ and $\mathbf{s}$ *we have*
$$\mathcal{V}^R\left(\zeta,\mathbf{q},\frac{\mathbf{r}}{\alpha},\frac{\mathbf{s}}{\alpha}\right) \overset{\alpha\to\infty}{\longrightarrow} \mathcal{V}^R_{\mathbf{r}}(\zeta,\mathbf{q}) + \mathcal{V}^R_{\mathbf{s}}(\zeta,\mathbf{q}) - \delta(\mathbf{q})e^{-2\beta\mathcal{C}_0(\mathbf{0})\zeta}. \tag{58}$$

Proof. In case (1), the rapid phases cancel the contributions of all but the last term in (54), and we get $\frac{\partial\mathcal{V}^R}{\partial\zeta} = -2\beta\mathcal{C}_0(\mathbf{0})\mathcal{V}^R$, which gives (55).

In case (2), we obtain in the limit $\alpha\to\infty$ the simplified system
$$\frac{\partial\mathcal{V}^R_{\mathbf{r}}}{\partial\zeta} = \frac{\beta}{(2\pi)^d}\int \widehat{\mathcal{C}}_0(\mathbf{u})\left[\mathcal{V}^R_{\mathbf{r}}\left(\zeta,\mathbf{q}-\frac{1}{2}\mathbf{u},\mathbf{s}-\mathbf{u}\right)e^{-i\mathbf{r}\cdot\mathbf{u}\zeta} + \mathcal{V}^R_{\mathbf{r}}\left(\zeta,\mathbf{q}-\frac{1}{2}\mathbf{u},\mathbf{s}+\mathbf{u}\right)e^{i\mathbf{r}\cdot\mathbf{u}\zeta} - 2\mathcal{V}^R_{\mathbf{r}}(\zeta,\mathbf{q},\mathbf{s})\right]d\mathbf{u}.$$
We then Fourier transforms this equation in $\mathbf{q}$ and $\mathbf{s}$, and obtain that the solution does not depend on $\mathbf{s}$, that it satisfies (56), and that it is given by (57).

In case (3) we obtain the simplified system for $\mathcal{V}^R_{\mathbf{r},\mathbf{s}}(\zeta,\mathbf{q}) = \lim_{\alpha\to\infty}\mathcal{V}^R(\zeta,\mathbf{q},\frac{\mathbf{r}}{\alpha},\frac{\mathbf{s}}{\alpha})$:
$$\frac{\partial\mathcal{V}^R_{\mathbf{r},\mathbf{s}}}{\partial\zeta} = \frac{2\beta}{(2\pi)^d}\int \widehat{\mathcal{C}}_0(\mathbf{u})\left[\mathcal{V}^R_{\mathbf{s}}\left(\zeta,\mathbf{q}-\frac{1}{2}\mathbf{u}\right)\cos(\mathbf{s}\cdot\mathbf{u}\zeta) + \mathcal{V}^R_{\mathbf{r}}\left(\zeta,\mathbf{q}-\frac{1}{2}\mathbf{u}\right)\cos(\mathbf{r}\cdot\mathbf{u}\zeta) - \mathcal{V}^R_{\mathbf{r},\mathbf{s}}(\zeta,\mathbf{q})\right]d\mathbf{u}.$$
Using Eq. (56) satisfied by $\mathcal{V}^R_{\mathbf{s}}$ and $\mathcal{V}^R_{\mathbf{r}}$, we get
$$\frac{\partial\mathcal{V}^R_{\mathbf{r},\mathbf{s}}}{\partial\zeta} = \frac{\partial\mathcal{V}^R_{\mathbf{r}}}{\partial\zeta} + \frac{\partial\mathcal{V}^R_{\mathbf{s}}}{\partial\zeta} + 2\beta\mathcal{C}_0(\mathbf{0})\left[\mathcal{V}^R_{\mathbf{r}} + \mathcal{V}^R_{\mathbf{s}} - \mathcal{V}^R_{\mathbf{r},\mathbf{s}}\right],$$
which yields (58). □

The parameter β characterizes the loss of coherence: if $\beta \ll 1$, then the random medium has no influence on the propagation; if $\beta \gg 1$, then scattering is strong. If we assume that $\beta \gg 1$, then the function $\mathcal{V}^R_{\mathbf{r}}(\zeta,\mathbf{q})$ has the Gaussian form
$$\mathcal{V}^R_{\mathbf{r}}(\zeta,\mathbf{q}) \overset{\beta\gg1}{\simeq} (\pi\mathcal{D}\beta\zeta)^{-d/2}e^{-\frac{1}{\mathcal{D}\beta\zeta}|\mathbf{q}|^2 - \frac{\mathcal{D}\beta\zeta^3}{3}|\mathbf{r}|^2},$$
where $\mathcal{D}$ is defined by (46).

7. Analysis of the transmitted and reflected waves for slow transverse fluctuations

In the next two sections we discuss important consequences of the effective systems of transport equations for the transmission and reflection operators. We consider first, in this section, the case with slow transverse fluctuations ($l_x \gg r_0$). As mentioned in Section 1, the situation analyzed in this paper in which a wave propagates in a random medium, is reflected by an interface, and then propagates back in the same medium, can be encountered in many situations of interest, and in particular in optical coherence tomography. The problem has been analyzed in [18–20], where it is assumed that the statistics of the forward- and backward-propagating waves are *independent*. One of the main goals of the next two sections is to identify the regimes in which this approximation, the so-called independent approach, is valid. We will stress in this section that it is important to take into account that the waves propagate in the *same* medium in the regime in which the transverse correlation radius is larger than the beam width, because the predictions of the "rigorous" approach and the independent approach are quantitatively different for the reflected intensity profiles and qualitatively very different for the spatial autocorrelation function.

We assume that

(a) the reflectivity function of the interface at $z=0$ is constant,
(b) the pulse has carrier frequency k_0 and it is narrowband in the sense that it satisfies (41),
(c) the input beam spatial profile is Gaussian with radius r_0,

$$b_{\rm inc}(t,\mathbf{x}) = f(t)e^{-ik_0t}e^{-\frac{|\mathbf{x}|^2}{r_0^2}} + cc$$

(d) the transverse correlation radius l_x of the random fluctuations of the medium is larger than r_0 and $k_0r_0^2 \sim L$.

7.1. Statistics of the transmitted wave

As stated in Proposition 1, the transmitted wave $p^{\varepsilon}_{\rm tr}(s,\mathbf{x})$ in the plane $z=0$ converges to the random field $p_{\rm tr}(s,\mathbf{x})$ given by (18) where

$$\check{b}_{\rm inc}(k,\mathbf{x}) = \hat{f}(k-k_0)e^{-\frac{|\mathbf{x}|^2}{r_0^2}}, \quad \hat{f}(k) = \int f(t)e^{ikt}\,dt.$$

In a homogeneous medium, the transmitted wave has the Gaussian form

$$p_{\rm tr,homo}(s,\mathbf{x}) = T_0 \frac{e^{-i\frac{d}{2}\operatorname{atan}\left(\frac{2L}{k_0r_0^2}\right)}}{\left(1+\frac{4L^2}{k_0^2r_0^4}\right)^{d/4}} \exp\left[-\frac{|\mathbf{x}|^2}{r_0^2\left(1+\frac{4L^2}{k_0^2r_0^4}\right)} + i\frac{|\mathbf{x}|^2}{r_0^2}\frac{\frac{2L}{k_0r_0^2}}{1+\frac{4L^2}{k_0^2r_0^4}}\right] f(s)e^{-ik_0s} + cc,$$

which exhibits the usual diffractive spreading.

In a random medium the coherent (i.e. mean) transmitted wave is

$$\mathbb{E}[p_{\rm tr}(s,\mathbf{x})] = \exp\left[-\frac{[C_{2k_0}(\mathbf{0})+C_0(\mathbf{0})]k_0^2}{8}L\right]p_{\rm tr,homo}(s,\mathbf{x}), \tag{59}$$

which exhibits a strong damping, but the shape is not affected compared to the homogeneous case. In fact, the wave becomes incoherent and its statistical properties are captured by its second-order statistics. The autocorrelation function of the transmitted wave is

$$A_{\rm tr}(s,s',\mathbf{x},\mathbf{x}') = \mathbb{E}[p_{\rm tr}(s,\mathbf{x})p_{\rm tr}(s',\mathbf{x}')],$$

and it can be expressed in terms of the Wigner distribution as

$$A_{\rm tr}(s,s',\mathbf{x},\mathbf{x}') = \exp\left[-\frac{C_{2k_0}(\mathbf{0})k_0^2}{4}L\right]f(s)f(s')e^{ik_0(s'-s)}\frac{r_0^d}{(2\pi)^{3d/2}}\int\int\int W^T\left(L,\frac{\mathbf{x}+\mathbf{x}'}{2},\mathbf{x}'',\mathbf{q},\mathbf{q}'\right)e^{-i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}e^{-\frac{2|\mathbf{x}''|^2}{r_0^2}-\frac{|\mathbf{q}'|^2r_0^2}{2}}d\mathbf{x}''d\mathbf{q}d\mathbf{q}' + cc.$$

The results of this subsection so far are valid under the assumptions (a)–(c) and that we now specialize to the slow transversal variation case by assuming also (d). Using the expression (48) of the Wigner distribution, we obtain

$$A_{\rm tr}(s,s',\mathbf{x},\mathbf{x}') = T_0^2\exp[-q_T(L)]f(s)f(s')e^{ik_0(s'-s)}\left(\frac{r_0}{r_T(L)}\right)^d \exp\left(-\frac{|\mathbf{x}|^2+|\mathbf{x}'|^2}{r_T(L)^2} - \frac{|\mathbf{x}-\mathbf{x}'|^2}{\rho_T^2(L)} + i\chi_T(L)\frac{|\mathbf{x}|^2-|\mathbf{x}'|^2}{r_T^2(L)}\right) + cc, \tag{60}$$

where

$$q_T(L) = \frac{C_{2k_0}(\mathbf{0})k_0^2}{4}L, \tag{61}$$

$$\chi_T(L) = \frac{2L}{k_0r_0^2} + \frac{k_0DL^2}{4}, \tag{62}$$

$$r_T(L) = r_0\left(1+\frac{4L^2}{k_0^2r_0^4}+\frac{DL^3}{3r_0^2}\right)^{1/2}, \tag{63}$$

$$\rho_T(L) = r_T(L)\left(\frac{k_0^2r_0^2DL}{8}+\frac{DL^3}{6r_0^2}+\frac{k_0^2D^2L^4}{96}\right)^{-1/2}. \tag{64}$$

The factor $q_T(L)$ given by (61) gives the damping of the autocorrelation function. By comparing with the damping factor of the coherent wave, one finds that the transmitted wave is essentially incoherent in the situation with $C_{2k_0}(\mathbf{0}) \ll C_0(\mathbf{0})$. The damping factor (61) expresses the loss due to incoherent backscattering, while the damping factor (59) expresses the loss due to incoherent forward and backward scattering.

The beam radius $r_T(L)$ is given by (63). If $DLk_0^2r_0^2 \ll 1$, then the random component is negligible and we obtain the usual formula for the paraxial beam spreading. If $DLk_0^2r_0^2 \gg 1$, then the random component is dominant and the beam radius increases as $L^{3/2}$. This result is well known, it was obtained in the physical literature in Ref. [8] and confirmed mathematically for instance in Ref. [6].

The correlation radius of the beam $\rho_T(L)$ is given by (64). If $DLk_0^2r_0^2 \ll 1$, then the correlation radius is much larger than r_0, which simply means that randomness plays no role. If $DLk_0^2r_0^2 \gg 1$, then the correlation radius decays as $L^{-1/2}$. This asymptotic result can also be found in [6].

7.2. Statistics of the reflected wave

As stated in Proposition 2, the reflected wave $p^\varepsilon_{\mathrm{ref}}(s,\mathbf{x})$ converges to the random field $p_{\mathrm{ref}}(s,\mathbf{x})$ given by (32). In a homogeneous medium, the reflected wave has the Gaussian form

$$p_{\mathrm{ref,homo}}(s,\mathbf{x}) = R_0 \frac{e^{-i\frac{d}{2}\mathrm{atan}\left(\frac{4L}{k_0r_0^2}\right)}}{\left(1+\frac{16L^2}{k_0^2r_0^4}\right)^{d/4}} \exp\left[-\frac{|\mathbf{x}|^2}{r_0^2\left(1+\frac{16L^2}{k_0^2r_0^4}\right)} + i\frac{|\mathbf{x}|^2}{r_0^2}\frac{\frac{4L}{k_0r_0^2}}{1+\frac{16L^2}{k_0^2r_0^4}}\right] f(s)e^{-ik_0s} + cc,$$

which can be obtained from the expression of the homogeneous transmitted field simply by substituting L by $2L$ (and T_0 by R_0). In a random medium the coherent reflected wave is

$$\mathbb{E}[p_{\mathrm{ref}}(s,\mathbf{x})] = \exp\left[-\frac{[C_{2k_0}(\mathbf{0})+C_0(\mathbf{0})]k_0^2}{4}L\right] p_{\mathrm{ref,homo}}(s,\mathbf{x}), \tag{65}$$

which exhibits an exponential damping with a damping rate multiplied by two compared to the one of the transmitted wave. Eq. (65) is valid under the assumptions (a)–(c) and we now specialize to the slow transversal variation case by assuming also (d). As in the case of the transmitted wave, the reflected wave is essentially incoherent. The autocorrelation function of the reflected wave is

$$A_{\mathrm{ref}}(s,s',\mathbf{x},\mathbf{x}') = \mathbb{E}[p_{\mathrm{ref}}(s,\mathbf{x})p_{\mathrm{ref}}(s',\mathbf{x}')].$$

Using the expression (53) of the Wigner distribution, we obtain

$$A_{\mathrm{ref}}(s,s',\mathbf{x},\mathbf{x}') = R_0^2 \exp[-q_R(L)] f(s)f(s')e^{ik_0(s'-s)}\left(\frac{r_0}{r_R(L)}\right)^d \exp\left(-\frac{|\mathbf{x}|^2+|\mathbf{x}'|^2}{r_R^2(L)} - \frac{|\mathbf{x}-\mathbf{x}'|^2}{\rho_R^2(L)} + i\chi_R(L)\frac{|\mathbf{x}|^2-|\mathbf{x}'|^2}{r_R^2(L)}\right) + cc, \tag{66}$$

where

$$q_R(L) = \frac{C_{2k_0}(\mathbf{0})k_0^2}{2}L, \tag{67}$$

$$\chi_R(L) = \frac{4L}{k_0r_0^2} + 2k_0DL^2, \tag{68}$$

$$r_R(L) = r_0\left(1+\frac{16L^2}{k_0^2r_0^4}+\frac{4DL^3}{r_0^2}\right)^{1/2}, \tag{69}$$

$$\rho_R(L) = r_R(L)\left(\frac{k_0^2r_0^2DL}{2}+\frac{2DL^3}{r_0^2}\right)^{-1/2}. \tag{70}$$

The beam radius $r_R(L)$ is given by (69). Qualitatively, the beam spreading is of the same type as the one of the transmitted wave, but quantitatively, it is enhanced compared to a propagation through a random medium with length $2L$ (see below for a quantitative comparison).

The correlation radius of the beam $\rho_R(L)$ is given by (70). If $DLk_0^2r_0^2 \ll 1$, then the correlation radius is very large, which simply means that the reflected wave is still coherent. More surprisingly, if $DLk_0^2r_0^2 \gg 1$, then the correlation radius goes to the constant value $\sqrt{2}r_0$. This result is in contrast with the one obtained for the transmitted wave, whose correlation radius decays as $L^{-1/2}$. This means that the wave looses its coherence as it propagates deep into the random medium, but it recovers part of it when it is reflected. This fact is related to the special case addressed here in which the lateral variations of the random medium are very slow. However, this configuration can be encountered for instance when addressing laser propagation in the atmosphere or acoustic/elastic waves in the earth crust.

Let us compare these results with the ones that we can obtain using the simple and naive approach in which we assume that the statistics of the forward- and backward-propagating waves are independent. Within this approach, the intensity profile and autocorrelation function of the reflected wave in the plane $z = L$ are given by (60) where L should be replaced by $2L$ (and the overall multiplicative factor R_0^2/T_0^2 should be applied). We therefore see that the beam spreading is slightly underestimated by the independent approach, which predicts that

$$r_R(L)|_{\text{ind}} = r_0\sqrt{1 + \frac{16L^2}{k_0^2 r_0^4} + \frac{8DL^3}{3r_0^2}}, \tag{71}$$

while the last term should have a factor 4 instead of 8/3 according to the exact formula (69). This comes from the fact that the wave revisits the same perturbations when propagating back in the same random medium, and it is well known that (for instance) $\mathbb{E}[(B_L + B_L')^2] < \mathbb{E}[(2B_L)^2]$ for two independent Brownian motions B_L and B_L'. The independent approach does not capture the correct coherence properties of the reflected wave either, since it predicts that the correlation radius should be

$$\rho_R(L)|_{\text{ind}} = r_0 \frac{\sqrt{1 + \frac{16L^2}{k_0^2 r_0^4} + \frac{8DL^3}{3r_0^2}}}{\sqrt{\frac{k_0^2 r_0^2 DL}{4} + \frac{4DL^3}{3r_0^2} + \frac{k_0^2 D^2 L^4}{6}}} \overset{DLk_0^2 r_0^2 \gg 1}{\simeq} \frac{4}{k_0\sqrt{DL}}, \tag{72}$$

while it should converge to $\sqrt{2}r_0$ as $DLk_0^2r_0^2 \gg 1$ according to (70), a qualitatively different result.

7.3. The reflected wave for a diffusive mirror

In the previous sections we considered the (standard) case of a specular reflection at the interface $z = 0$. For applications, it is important to discuss the case of diffuse backscattering. For instance, in optical coherence tomography, it is diffuse backscattering that actually occurs in the case of (skin) tissue [19]. In this subsection, we revisit the theory in the case in which an inhomogeneous mirror is inserted in the plane $z = 0$, with the impedance $Z_M(\mathbf{x})$, so that the second boundary condition in (9) now reads

$$\check{a}^\varepsilon(k, 0, \mathbf{x}) = R_M(\mathbf{x})\check{b}^\varepsilon(k, 0, \mathbf{x}),$$

where $R_M(\mathbf{x}) = (Z_M(\mathbf{x}) - 1)/(Z_M(\mathbf{x}) + 1)$ is the local reflection coefficient of the mirror. In this case, the initial condition at the reflecting interface $z = 0$ for the reflection operator is

$$\check{\mathcal{R}}^\varepsilon(k, 0, \mathbf{x}, \mathbf{x}') = R_M(\mathbf{x})\delta(\mathbf{x} - \mathbf{x}').$$

We shall assume here that R_M is a stationary random process, with mean zero and autocorrelation function

$$\mathbb{E}[R_M(\mathbf{x})\overline{R_M}(\mathbf{x}')] = R_0^2\psi(\mathbf{x} - \mathbf{x}').$$

Under these conditions, the initial condition for the Wigner distribution is $W^R(z = 0, \mathbf{x}, \mathbf{x}', \mathbf{q}, \mathbf{q}') = R_0^2\delta(\mathbf{x} - \mathbf{x}')\hat{\psi}(\mathbf{q} + \mathbf{q}')$. Assuming that ψ is Gaussian:

$$\psi(\mathbf{x}) = e^{-\frac{|\mathbf{x}|^2}{a^2}}, \tag{73}$$

where a is the correlation radius of the diffusive mirror, we then find by integrating (52) that

$$W^R(z, \mathbf{x}, \mathbf{x}', \mathbf{q}, \mathbf{q}') = R_0^2\left(\frac{2\pi}{k^2Dz + \frac{2}{a^2}}\right)^{d/2} \delta\left(\mathbf{x} - \mathbf{x}' - \frac{\mathbf{q} - \mathbf{q}'}{k}z\right) e^{-\frac{|\mathbf{q}+\mathbf{q}'|^2}{2k^2Dz + \frac{4}{a^2}}}.$$

By taking the limit $a \to \infty$, we recover the result obtained with a standard mirror with specular reflection.

If the input beam is Gaussian with carrier wavenumber k_0 and radius r_0, then the autocorrelation function of the reflected wave has the form (66) where the parameter $\chi_R(L)$, the beam width $r_R(L)$, and the correlation radius $\rho_R(L)$ are now given by

$$\chi_R(L) = \frac{4L}{k_0}\left(\frac{1}{r_0^2} + \frac{1}{a^2}\right) + 2k_0DL^2,$$

$$r_R(L) = r_0\left(1 + \frac{8L^2}{k_0^2 r_0^2}\left(\frac{2}{r_0^2} + \frac{1}{a^2}\right) + \frac{4DL^3}{r_0^2}\right)^{1/2},$$

$$\rho_R(L) = r_R(L)\left(\frac{r_0^2}{a^2}\left(1 + \frac{4L^2}{k_0^2 r_0^4}\right) + \frac{k_0^2 r_0^2 DL}{2} + \frac{2DL^3}{r_0^2}\right)^{-1/2}.$$

The presence of a diffusive mirror in place of a specular mirror affects both the beam radius and the correlation radius. In particular, the correlation radius is not infinite but equal to a for $L \to 0$. However, the long-distance behavior is the same as in the case of a specular mirror, and we confirm the previous result that the correlation radius is approximately $\sqrt{2}r_0$ when random scattering is strong.

7.4. Numerical simulations

One of the most striking results obtained in this section is that, compared to the case where one assumes that the forward and backward propagations are independent, the fact that the wave visits the same medium in the forward and in the backward propagation induces an increase of the beam width, as can be seen by comparing (69) and (71), and an increase of the correlation radius, as can be seen by comparing (70) and (72). We have performed numerical simulations to illustrate and confirm these predictions.

The numerical simulations are performed in the paraxial regime with a one-dimensional transverse space. We assume the presence of a perfectly reflecting mirror at $z = 0$ and a Gaussian input beam with carrier wavenumber $k_0 = 1$ and radius $r_0 = 4$ at $z = L = 10$. The random medium is modeled by a Gaussian process with Gaussian autocorrelation function with transverse correlation radius $l_c = 80$, longitudinal correlation length 1, and standard deviation $30/\sqrt{\pi}$. These parameters are at the border of our theoretical regime, but allow for easy numerical simulations. Here $D = 0.28$. We use a split-step Fourier method for discretizing the wave propagation. Finally, we perform a series of 1000 independent simulations to extract the mean intensity profiles $x \mapsto \mathbb{E}[p_{\text{ref}}(x)^2]$ and the autocorrelation function $x \mapsto \mathbb{E}[p_{\text{ref}}(x)p_{\text{ref}}(0)]$.

We simulate the backward propagation either with the same random medium as during the forward propagation (Fig. 2), or with an independent medium (Fig. 3). Then we compare the numerically averaged intensity profiles and autocorrelation functions with the theoretical formulas obtained above, which gives excellent agreement. In particular, in Fig. 2, one can

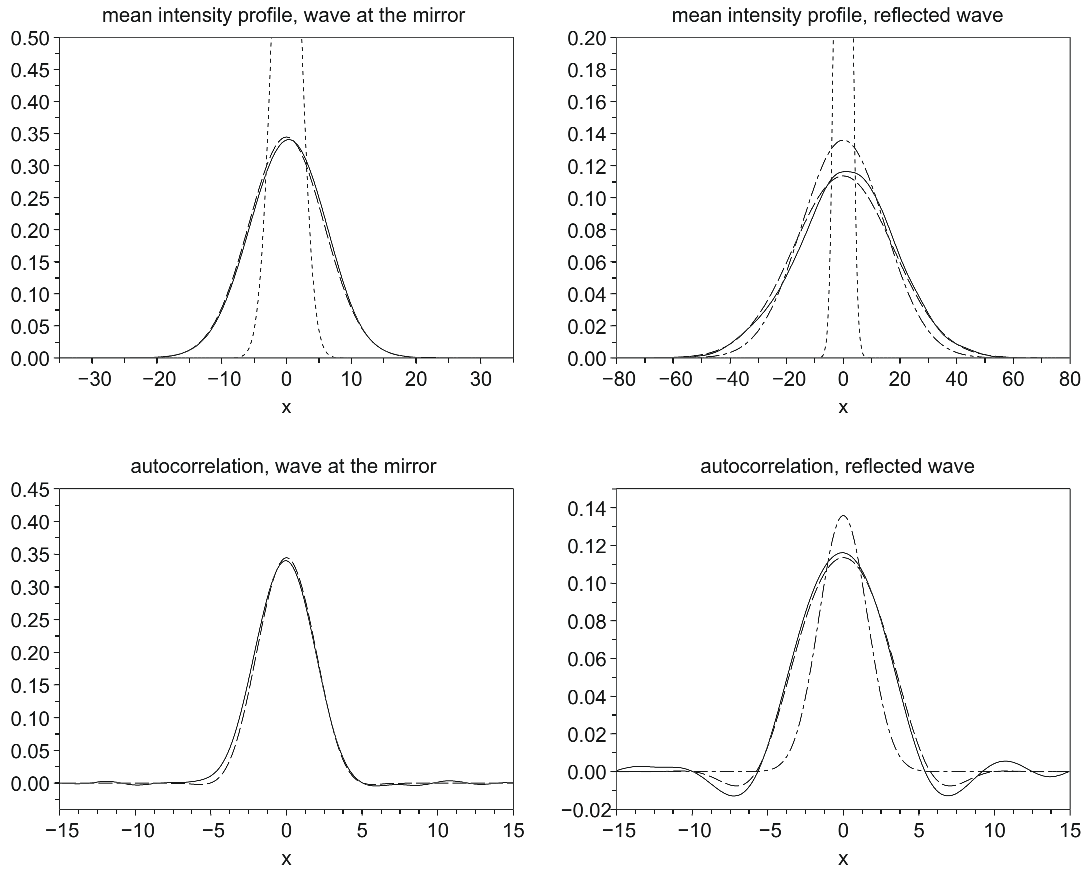

Fig. 2. Mean intensity profiles and autocorrelation functions of the wave at the mirror $z = 0$ (left) and of the reflected wave at $z = L$ (right). The dotted lines represent the intensity profile of the input beam. The solid lines are the results of the numerical simulations. The dashed lines are the theoretical formulas predicted by the rigorous theory that takes into account that the medium is the same in the forward and in the backward propagations. The dot-dashed lines are the theoretical formulas predicted by the independent approach. Here, in the numerical simulations, the medium is the same in the forward and in the backward propagations (i.e. it is the real situation). One can check that the numerical results are in agreement with the theoretical formulas predicted by the rigorous approach.

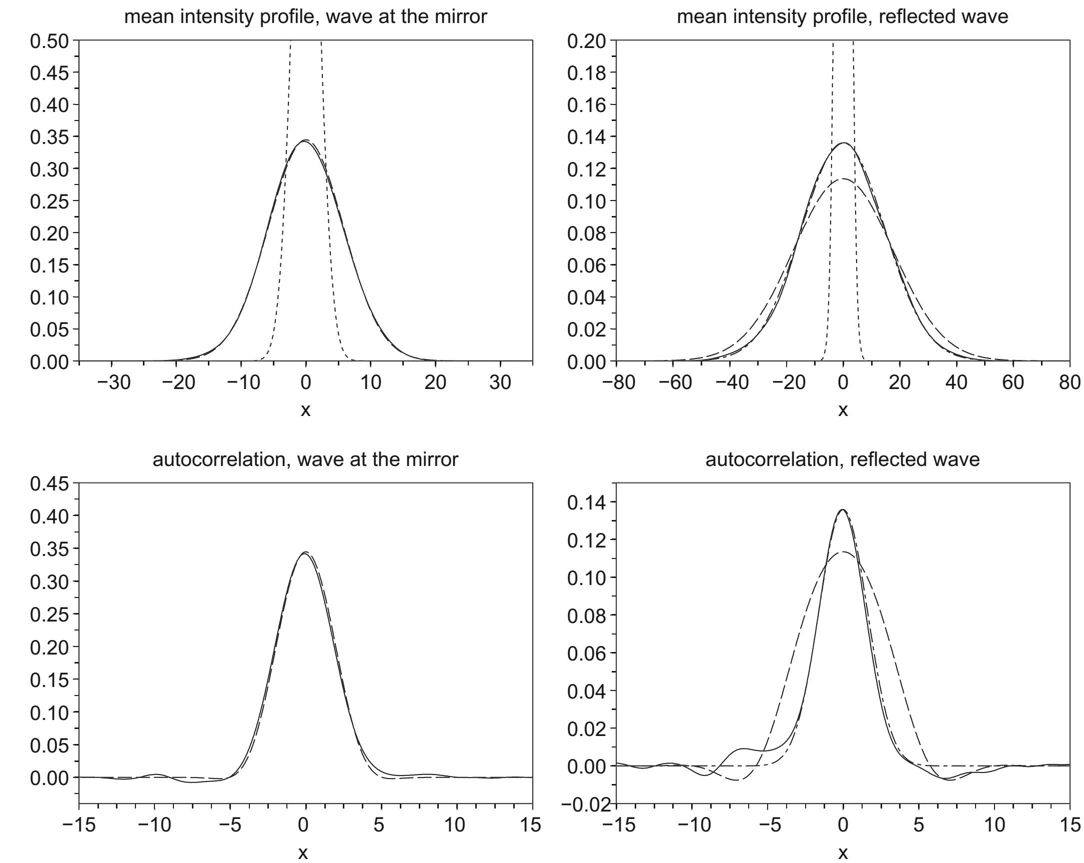

Fig. 3. The same as in Fig. 2, but here, in the numerical simulations, the medium is different in the forward and in the backward propagations (i.e. it is an artificial situation). One can check that the numerical results are in agreement with the theoretical formulas predicted by the independent approach.

check that the correlation radius of the reflected beam is larger than the correlation radius of the wave that reaches the mirror: when propagating back, the wave continues to spread out, but it recovers some coherence.

8. Analysis of the transmitted and reflected waves for rapid transverse fluctuations

In this section we assume the hypotheses (a)–(c) of Section 7, but instead of (d) we assume that the beam width r_0 is such that $r_0 \gg l_x$ and $k_0 r_0 l_x \sim L$.

We will show that, in this regime, the independent approach essentially gives the correct answer. The only important phenomenon not captured by the independent approach is the enhanced backscattering phenomenon, which we discuss in Section 8.3.

8.1. Second-order statistics of the transmitted wave

The autocorrelation function of the transmitted wave is given by

$$A_{\mathrm{tr}}(s,s',\mathbf{x},\mathbf{x}') = T_0^2 \exp[-q_T(L)] f(s) f(s') e^{ik_0(s'-s)} \left(\frac{r_0^2}{8\pi\alpha^2 l_x^2}\right)^{d/2} \int e^{-\frac{l_x^2}{2r_0^2}\left|\mathbf{s}+\frac{\mathbf{x}-\mathbf{x}'}{l_x}\right|^2 - \frac{r_0^2}{8\alpha^2 l_x^2}|\mathbf{s}|^2} e^{-i\mathbf{s}\cdot\frac{\mathbf{x}+\mathbf{x}'}{2\alpha l_x}} e^{\beta \int_0^1 \mathcal{C}_0\left(\mathbf{s}\zeta + \frac{\mathbf{x}-\mathbf{x}'}{l_x}\right) - \mathcal{C}_0(\mathbf{0})\, d\zeta} d\mathbf{s} + cc. \tag{74}$$

This expression is valid for any values of α and β. If $\alpha \gg 1$ and $\alpha \sim r_0/l_x$ (equivalently $l_x \ll r_0$ and $k_0 r_0 l_x \sim L$), then we have to leading order

$$A_{\mathrm{tr}}(s,s',\mathbf{x},\mathbf{x}') = T_0^2 \exp[-q_T(L)] f(s) f(s') e^{ik_0(s'-s)} \left(\frac{r_0^2}{8\pi\alpha^2 l_x^2}\right)^{d/2} \int e^{-\frac{r_0^2}{8\alpha^2 l_x^2}|\mathbf{s}|^2} e^{-\frac{|\mathbf{x}-\mathbf{x}'|^2}{2r_0^2}} e^{-i\mathbf{s}\cdot\frac{\mathbf{x}+\mathbf{x}'}{2\alpha l_x}} e^{\beta \int_0^1 \mathcal{C}_0\left(\mathbf{s}\zeta + \frac{\mathbf{x}-\mathbf{x}'}{l_x}\right) - \mathcal{C}_0(\mathbf{0})\, d\zeta} d\mathbf{s} + cc. \tag{75}$$

If moreover, $\beta \gg 1$, then we obtain that the autocorrelation function has the Gaussian shape given by (60) with

$$r_T(L) = r_0\left(1+\frac{4}{3}\mathcal{D}\beta\frac{\alpha^2 l_x^2}{r_0^2}\right)^{1/2} = r_0\left(1+\frac{DL^3}{3r_0^2}\right)^{1/2}, \tag{76}$$

$$\rho_T(L) = r_T(L)\left(\frac{\beta\mathcal{D}r_0^2}{2l_x^2}+\frac{\beta^2\mathcal{D}^2\alpha^2}{6}\right)^{-1/2} = r_T(L)\left(\frac{k_0^2 r_0^2 DL}{8}+\frac{k_0^2 D^2 L^4}{96}\right)^{-1/2}, \tag{77}$$

$$\chi_T(L) = \beta\mathcal{D}\alpha = \frac{k_0 DL^2}{4}. \tag{78}$$

As noted in Section 5.2, the large β-behavior of the transmitted wave is independent of α. This is why we find the same result as in Section 7.1 (see (63) with $k_0 r_0^2 \gg L$).

8.2. Second-order statistics of the reflected wave

Using the function $\mathcal{V}^R$, we find the following integral representation for the autocorrelation function of the reflected wave:

$$A_{\mathrm{ref}}(s,s',\mathbf{x},\mathbf{x}') = R_0^2\exp[-q_R(L)]f(s)f(s')e^{ik_0(s'-s)}\left(\frac{r_0^2}{8\pi l_x^2}\right)^d\int\int\int e^{-\frac{r_0^2}{8l_x^2}[|\mathbf{s}|^2+|\mathbf{r}-2\mathbf{q}|^2]+i\alpha\mathbf{r}\cdot\mathbf{s}-i\mathbf{s}\cdot\frac{\mathbf{x}+\mathbf{x}'}{2l_x}+i\frac{\mathbf{x}-\mathbf{x}'}{l_x}\cdot(\mathbf{q}+\frac{\mathbf{r}}{2})}\mathcal{V}^R(1,\mathbf{q},\mathbf{r},\mathbf{s})\,d\mathbf{q}\,d\mathbf{r}\,d\mathbf{s}+cc, \tag{79}$$

which holds true for any values of α and β. Using the asymptotic results of Lemma 2 (second item) we get that, in the regime $\alpha \gg 1$:

$$A_{\mathrm{ref}}(s,s',\mathbf{x},\mathbf{x}') = R_0^2\exp[-q_R(L)]f(s)f(s')e^{ik_0(s'-s)}\left(\frac{r_0^2}{32\pi\alpha^2 l_x^2}\right)^{d/2}\int e^{-\frac{r_0^2}{32\alpha^2 l_x^2}|\mathbf{s}|^2-\frac{|\mathbf{x}-\mathbf{x}'|^2}{2r_0^2}-i\mathbf{s}\cdot\frac{\mathbf{x}+\mathbf{x}'}{4\alpha l_x}}e^{2\beta\int_0^1 \mathcal{C}_0\left(\frac{\mathbf{x}-\mathbf{x}'}{l_x}+\mathbf{s}\zeta\right)-\mathcal{C}_0(\mathbf{0})d\zeta}d\mathbf{s}+cc. \tag{80}$$

This is the form (75) of the autocorrelation function of the transmitted wave, upon the substitution 2α for α and 2β for β. Therefore, the autocorrelation function of the reflected wave has the form of the autocorrelation function of the transmitted wave for a propagation distance $2L$. This shows that we would have obtained the same result if we had assumed that the backward propagation was independent of the forward propagation. The independent approach is valid in the regime $\alpha \gg 1$, but not in the regime $\alpha \ll 1$ as we have seen in Section 7, nor in the regime $\alpha \sim 1$ as can be seen by comparing the full expressions (74) and (79).

If we consider a diffusive mirror with the autocorrelation $\psi(\mathbf{x})$ as introduced in Section 7.3, then the autocorrelation function of the reflected field is given by (79) where $\mathcal{V}^R$ is the solution of the system (54) with the initial condition $\mathcal{V}^R(\zeta=0,\mathbf{q},\mathbf{r},\mathbf{s}) = (\pi l_x)^{-d}\hat{\psi}(2\mathbf{q}/l_x)$. In the limit $\alpha \to \infty$, we find that the function $\mathcal{V}^R(\zeta,\mathbf{q},\mathbf{r},\frac{\mathbf{s}}{\alpha})$ for any $\mathbf{r}\neq\mathbf{0}$ converges to

$$\mathcal{V}^R_{\mathbf{r}}(\zeta,\mathbf{q}) = \frac{1}{(2\pi)^d}\int\psi\left(\frac{l_x\mathbf{u}}{2}\right)e^{-i\mathbf{q}\cdot\mathbf{u}}e^{\beta\int_{-\zeta}^{\zeta}\mathcal{C}_0\left(\frac{\mathbf{u}}{2}+\mathbf{r}\zeta'\right)-\mathcal{C}_0(\mathbf{0})d\zeta'}d\mathbf{u}. \tag{81}$$

As a result we get that, in the regime $\alpha \gg 1$:

$$A_{\mathrm{ref}}(s,s',\mathbf{x},\mathbf{x}') = R_0^2\exp[-q_R(L)]f(s)f(s')e^{ik_0(s'-s)}\left(\frac{r_0^2}{32\pi\alpha^2 l_x^2}\right)^{d/2}\int\psi\left(\frac{l_x\mathbf{s}}{2}+\mathbf{x}-\mathbf{x}'\right)e^{-\frac{r_0^2}{32\alpha^2 l_x^2}|\mathbf{s}|^2-\frac{|\mathbf{x}-\mathbf{x}'|^2}{2r_0^2}-i\mathbf{s}\cdot\frac{\mathbf{x}+\mathbf{x}'}{4\alpha l_x}}e^{2\beta\int_0^1\mathcal{C}_0\left(\frac{\mathbf{x}-\mathbf{x}'}{l_x}+\mathbf{s}\zeta\right)-\mathcal{C}_0(\mathbf{0})\,d\zeta}d\mathbf{s}+cc.$$

If, additionally, we assume that $\beta \gg 1$ and the function ψ is Gaussian with radius a as in (73), then the autocorrelation function of the reflected wave has the Gaussian shape given by (66) in which

$$r_R(L) = r_0\left(1+\frac{8DL^3}{3r_0^2}+\frac{8L^2}{k_0^2a^2r_0^2}\right)^{1/2},$$

$$\rho_R(L) = r_R(L)\left(\frac{k_0^2r_0^2DL}{4}+\frac{k_0^2D^2L^4}{6}+\frac{2DL^3}{3a^2}+\frac{r_0^2}{a^2}\right)^{-1/2},$$

$$\chi_R(L) = k_0DL^2+\frac{4L}{k_0a^2}.$$

The presence of a diffusive mirror in place of a specular mirror affects both the beam radius and the correlation radius. However, the large-distance behavior is the same as the case of the specular mirror.

8.3. Enhanced backscattering

The comparison of the autocorrelation function of the reflected wave and that of the transmitted wave for the propagation distance $2L$ shows that, in the regime $\alpha \gg 1$, there is no coherent effect building up between the forward and backward propagations. However, there are some effects in the corrective terms. In this section we show that the reflected intensity presents a singular picture in a very narrow cone, of angular width of order α^{-1}, around the backscattered direction. This phenomenon called enhanced backscattering or weak localization is widely discussed in the physical literature [2,16]. It is here analyzed in the situation with an incoming wave with a narrow wave vector spectrum, or a "quasi" plane-wave, and it then arises as a consequence of the multiscale behavior of the regime $\alpha \gg 1$ that is exhibited in Lemma 2.

In this section, we assume that the incoming wave has the form

$$b_{\rm inc}(t,\mathbf{x}) = f(t)e^{-ik_0 t}g_{\rm inc}(\mathbf{x}) + cc,$$

and that it is nearly a plane wave, in the sense that $\hat{g}_{\rm inc}(\boldsymbol{\kappa})$ is concentrated at some $\boldsymbol{\kappa}_{\rm inc}$, with an angular width smaller than α^{-1}. The reflected signal in the direction $\boldsymbol{\kappa}_0$ is

$$\check{p}^{\varepsilon}_{\rm ref}(s,\boldsymbol{\kappa}_0) = \int p^{\varepsilon}_{\rm ref}(s,\mathbf{x})e^{-i\boldsymbol{\kappa}_0\cdot\mathbf{x}}d\mathbf{x} = \frac{1}{2\pi}\int \check{\mathcal{R}}^{\varepsilon}(k,L,\boldsymbol{\kappa}_0,\boldsymbol{\kappa}')\hat{b}_{\rm inc}(k,\boldsymbol{\kappa}')e^{-iks}\,dk.$$

The moment of the square modulus of $\check{p}^{\varepsilon}_{\rm ref}(s,\boldsymbol{\kappa}_0)$ only involves specific moments of the form (35) (with distinct k). Therefore this moment converges to the one of the limit process $\check{p}_{\rm ref}(s,\boldsymbol{\kappa}_0)$ defined as the Fourier transform in $\mathbf{x}$ of $p_{\rm ref}(s,\mathbf{x})$ given by (32). This means that the mean reflected intensity in the direction $\boldsymbol{\kappa}_0$ satisfies

$$\mathbb{E}[|\check{p}^{\varepsilon}_{\rm ref}(s,\boldsymbol{\kappa}_0)|^2] \stackrel{\varepsilon\to 0}{\to} R_0^2\exp[-q_R(L)]f^2(s)I^R(\boldsymbol{\kappa}_0),$$
$$I^R(\boldsymbol{\kappa}_0) = 2^{-d}l_x^d\int \mathcal{V}^R\Big(1,\frac{\boldsymbol{\kappa}_0-\boldsymbol{\kappa}_1}{2}l_x,(\boldsymbol{\kappa}_0+\boldsymbol{\kappa}_1)l_x,\mathbf{0}\Big)|\hat{g}_{\rm inc}(\boldsymbol{\kappa}_1)|^2\,d\boldsymbol{\kappa}_1.$$

Using the fact that $\hat{g}_{\rm inc}(\boldsymbol{\kappa})$ is concentrated at $\boldsymbol{\kappa}_{\rm inc}$, we get

$$I^R(\boldsymbol{\kappa}_0) = P\mathcal{V}^R\Big(1,\frac{\boldsymbol{\kappa}_0-\boldsymbol{\kappa}_{\rm inc}}{2}l_x,(\boldsymbol{\kappa}_0+\boldsymbol{\kappa}_{\rm inc})l_x,\mathbf{0}\Big),$$

where $P = R_0^2 2^{-d}l_x^d\int|\hat{g}_{\rm inc}(\boldsymbol{\kappa}_1)|^2\,d\boldsymbol{\kappa}_1$.

If random scattering is weak $\beta \ll 1$, then we have the usual specular reflection

$$I^R(\boldsymbol{\kappa}_0)|_{\beta\ll 1} = P\delta\Big(\frac{\boldsymbol{\kappa}_0-\boldsymbol{\kappa}_{\rm inc}}{2}l_x\Big).$$

In the presence of random scattering, the specular reflection takes the form of a Dirac peak at $\boldsymbol{\kappa}_{\rm inc}$ with intensity $\exp(-2\beta\mathcal{C}_0(\mathbf{0}))$ and a diffusive cone centered at $\boldsymbol{\kappa}_{\rm inc}$. More exactly, far enough from the backscattered direction $-\boldsymbol{\kappa}_{\rm inc}$, the mean reflected intensity is

$$I^R(\boldsymbol{\kappa}_0) = P\mathcal{V}^R_{\mathbf{0}}\Big(1,\frac{\boldsymbol{\kappa}_0-\boldsymbol{\kappa}_{\rm inc}}{2}l_x\Big),\quad \text{for}\quad |\boldsymbol{\kappa}_0+\boldsymbol{\kappa}_{\rm inc}|l_x \gg \alpha^{-1} = Pe^{-2\beta\mathcal{C}_0(\mathbf{0})}\left[\delta\Big(\frac{\boldsymbol{\kappa}_0-\boldsymbol{\kappa}_{\rm inc}}{2}l_x\Big)+\frac{1}{(2\pi)^d}\int e^{-il_x\frac{\boldsymbol{\kappa}_0-\boldsymbol{\kappa}_{\rm inc}}{2}\cdot\mathbf{u}}\Big(e^{2\beta\mathcal{C}_0(\frac{\mathbf{u}}{2})}-1\Big)\,d\mathbf{u}\right].$$

In a narrow cone around the backscattered direction, the reflected intensity is locally larger:

$$I^R(-\boldsymbol{\kappa}_{\rm inc}+\alpha^{-1}\boldsymbol{\kappa}) = P\big[\mathcal{V}^R_{\mathbf{0}}(1,-\boldsymbol{\kappa}_{\rm inc}l_x)+\mathcal{V}^R_{\boldsymbol{\kappa}l_x}(1,-\boldsymbol{\kappa}_{\rm inc}l_x)-e^{-2\beta\mathcal{C}_0(\mathbf{0})}\delta(\boldsymbol{\kappa}_{\rm inc}l_x)\big]. \tag{82}$$

If we assume, moreover, that $\beta \gg 1$, then we have

$$I^R(\boldsymbol{\kappa}_0) = (P2^dl_x^{-d})\Delta\kappa_{\rm spec}^{-d}e^{-\frac{|\boldsymbol{\kappa}_0-\boldsymbol{\kappa}_{\rm inc}|^2}{\Delta\kappa^2_{\rm spec}}},\quad \text{for}\quad |\boldsymbol{\kappa}_0+\boldsymbol{\kappa}_{\rm inc}|l_x \gg \alpha^{-1}, \tag{83}$$

where the width of the diffusion cone around the specular direction $\boldsymbol{\kappa}_{\rm inc}$ is:

$$\Delta\kappa_{\rm spec} = \frac{2\sqrt{\mathcal{D}\beta}}{l_x} = \frac{\sqrt{\mathcal{D}}\sigma k_0\sqrt{Ll_z}}{l_x} = \sqrt{DL}k_0. \tag{84}$$

On the top of this broad cone, we have an arrow cone of relative maximum equal to 2 centered along the back scattered direction $-\boldsymbol{\kappa}_{\rm inc}$:

$$I^R(-\boldsymbol{\kappa}_{\rm inc}+\alpha^{-1}\boldsymbol{\kappa}) = (P2^dl_x^{-d})\Delta\kappa_{\rm spec}^{-d}e^{-\frac{|\boldsymbol{\kappa}_0-\boldsymbol{\kappa}_{\rm inc}|^2}{\Delta\kappa^2_{\rm spec}}}\Big[1+e^{-\frac{\mathcal{D}\beta}{3}|\boldsymbol{\kappa}|^2l_x^2}\Big]. \tag{85}$$

This shows that the width of the enhanced backscattering cone is:

$$\Delta\kappa_{\rm EBC} = \frac{\sqrt{3}}{l_x\sqrt{\mathcal{D}\beta}\alpha} = \frac{2\sqrt{3}l_x}{\sqrt{\mathcal{D}}\sigma\sqrt{l_zL^3}} = \frac{2\sqrt{3}}{\sqrt{DL^3}}. \tag{86}$$

Note that the angular width $\Delta\theta_{\rm EBC} = \Delta\kappa_{\rm EBC}/k_0$ of the cone is proportional to the wavelength, as predicted by physical arguments (diagrammatic expansions) [16].

If we consider a diffusive mirror with the autocorrelation $\psi(\mathbf{x})$ as introduced in Section 7.3, then the reflected intensity is still given by (82) in the regime $\alpha \gg 1$, with $\mathcal{V}^R_{\mathbf{r}}$ given by (81). If, additionally, we assume that $\beta \gg 1$ and the function ψ is Gaussian with radius a, then we find that the diffusion cone is increased by the presence of the diffusive mirror is (83) with the width of the diffusion cone around the specular direction $\boldsymbol{\kappa}_{\mathrm{inc}}$ given by

$$\Delta\kappa_{\mathrm{spec}} = \frac{2\sqrt{\mathcal{D}\beta + \frac{l_x^2}{a^2}}}{l_x} = \frac{\sqrt{\mathcal{D}\sigma^2 k_0^2 L l_z + \frac{4 l_x^2}{a^2}}}{l_x} = \sqrt{D L k_0^2 + \frac{4}{a^2}}.$$

However, the relative amplitude, the width, and the shape of the enhanced backscattering cone are not affected by the presence of the diffusive mirror and the mean reflected intensity around the backscattered direction $-\boldsymbol{\kappa}_{\mathrm{inc}}$ is still given by (85).

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Appendix A. Derivation of the transmission model

We want to compute the expectations in (27) and define

$$I_N^\varepsilon(z) = \prod_{j=1}^N \widehat{\mathcal{T}}^\varepsilon(k_j, z, \boldsymbol{\kappa}_j, \boldsymbol{\kappa}_j').$$

Using (14) we find

$$\frac{d}{dz} I_N^\varepsilon(z) = \sum_{j=1}^N \prod_{l=1\neq j}^N \widehat{\mathcal{T}}^\varepsilon(k_l, z, \boldsymbol{\kappa}_l, \boldsymbol{\kappa}_l') \bigg\{ \int \widehat{\mathcal{T}}^\varepsilon(k_j, z, \boldsymbol{\kappa}_j, \boldsymbol{\kappa}_a) \widehat{\mathcal{L}}^\varepsilon(k_j, z, \boldsymbol{\kappa}_a, \boldsymbol{\kappa}_j')\, d\boldsymbol{\kappa}_a + e^{\frac{2ik_jz}{\varepsilon^2}} \int \widehat{\mathcal{T}}^\varepsilon(k_j, z, \boldsymbol{\kappa}_j, \boldsymbol{\kappa}_a) \widehat{\mathcal{L}}^\varepsilon(k_j, z, \boldsymbol{\kappa}_a, \boldsymbol{\kappa}_b) \widehat{\mathcal{R}}^\varepsilon(k_j, z, \boldsymbol{\kappa}_b, \boldsymbol{\kappa}_j') d\boldsymbol{\kappa}_a\, d\boldsymbol{\kappa}_b \bigg\}. \tag{A.1}$$

We next apply the diffusion approximation to get transport equations for the moments, see [9] for background material on and related applications of the diffusion approximation. Observe that the random coefficients are rapidly fluctuating in view of (15). Those coefficients that are of order ε^{-1} are centered and fluctuate on the scale ε^2, moreover they are assumed to be rapidly mixing, giving a white-noise scaling situation. We can thus apply diffusion approximation results to obtain equations for the moments $\mathbb{E}[I_N^\varepsilon]$ in the limit $\varepsilon \to 0$:

$$\bar{I}_N(z) = \lim_{\varepsilon \to 0} \mathbb{E}[I_N^\varepsilon(z)].$$

We obtain from (A.1) that $\bar{I}_N$ solves a system of integro-differential equations

$$\frac{d}{dz}\bar{I}_N(z) = -\frac{i}{2}\left(\sum_{j=1}^N \frac{|\boldsymbol{\kappa}_j'|^2}{k_j}\right)\bar{I}_N(z) - \frac{1}{8}\sum_{j=1}^2 \mathcal{I}_{N,j}(z), \tag{A.2}$$

with the initial conditions $\bar{I}_N(k_1, \ldots, k_N, \boldsymbol{\kappa}_1, \ldots, \boldsymbol{\kappa}_N, \boldsymbol{\kappa}_1', \ldots, \boldsymbol{\kappa}_N', z = 0) = T_0^N \delta(\boldsymbol{\kappa}_1 - \boldsymbol{\kappa}_1') \cdots \delta(\boldsymbol{\kappa}_N - \boldsymbol{\kappa}_N')$. The first term to the right-hand side of (A.2) is the contributions of the deterministic diffractive terms in (15). We next discuss the particular forms of the source terms $\mathcal{I}_{N,j}$. We have

$$\mathcal{I}_{N,1}(z) = \left(\sum_{j=1}^N k_j^2 C_0(\mathbf{0})\right)\bar{I}_N(z) + \frac{1}{(2\pi)^d}\sum_{j=1}^N \sum_{l=1\neq j}^N k_j k_l \int \widehat{C}_0(\boldsymbol{\kappa}) \bar{I}_N(z; \boldsymbol{\kappa}_l' - \boldsymbol{\kappa}, \boldsymbol{\kappa}_j' + \boldsymbol{\kappa})\, d\boldsymbol{\kappa},$$

which comes from the interaction of the first term in the right-hand side of (A.1) with the dynamics for $\widehat{\mathcal{T}}^\varepsilon$ as given in (14) and where we only show the shifted arguments for $\bar{I}_N$. The second term in the right-hand side of (A.1) interacts with the dynamics for $\widehat{\mathcal{R}}^\varepsilon$ as given in (13) and gives

$$\mathcal{I}_{N,2}(z) = \left(\sum_{j=1}^N k_j^2 C_{2k_j}(\mathbf{0})\right)\bar{I}_N(z).$$

We can conclude that

$$\frac{d}{dz}\bar{I}_N(z) = -\frac{i}{2}\left(\sum_{j=1}^N \frac{|\boldsymbol{\kappa}_j'|^2}{k_j}\right)\bar{I}_N(z) - \frac{1}{8}\left(\sum_{j=1}^N k_j^2 (C_0(\mathbf{0}) + C_{2k_j}(\mathbf{0}))\right)\bar{I}_N(z) - \frac{1}{8(2\pi)^d}\sum_{j=1}^N \sum_{l=1\neq j}^N k_j k_l \int \widehat{C}_0(\boldsymbol{\kappa}) \bar{I}_N(z; \boldsymbol{\kappa}_l' - \boldsymbol{\kappa}, \boldsymbol{\kappa}_j' + \boldsymbol{\kappa})\, d\boldsymbol{\kappa}.$$

Using in particular the relation

$$\mathbb{E}\left[\int\int\int_0^{z_a}\int_0^{z_b}\lambda(s_a,\boldsymbol{\kappa}_a)\lambda(s_b,\boldsymbol{\kappa}_b)\,dB_{s_a}(\boldsymbol{\kappa}_a)\,dB_{s_b}(\boldsymbol{\kappa}_b)\,d\boldsymbol{\kappa}_a\,d\boldsymbol{\kappa}_b\right]=\int\int_0^{\min(z_a,z_b)}\mathbb{E}[\lambda(s,\boldsymbol{\kappa})\lambda(s,-\boldsymbol{\kappa})](2\pi)^d\widehat{C}_0(\boldsymbol{\kappa})\,ds\,d\boldsymbol{\kappa},$$

we can then verify that

$$\bar{I}_N(z)=\mathbb{E}\left[\prod_{j=1}^N\widehat{\mathcal{T}}(k_j,z,\boldsymbol{\kappa}_j,\boldsymbol{\kappa}_j')\right],$$

when the right-hand side expectation is taken with respect to the Itô–Schrödinger model for the transmission operator in (28).

Appendix B. Derivation of wave pulse reflection model

We now want to compute the expectations in (35) and define

$$J_N^\varepsilon(z)=\prod_{j=1}^N\widehat{\mathcal{R}}^\varepsilon(k_j,z,\boldsymbol{\kappa}_j,\boldsymbol{\kappa}_j').$$

Using (13) we find

$$\frac{d}{dz}J_N^\varepsilon(z)=\sum_{j=1}^N\prod_{l=1\neq j}^N\widehat{\mathcal{R}}^\varepsilon(k_l,z,\boldsymbol{\kappa}_l,\boldsymbol{\kappa}_l')\left\{e^{-\frac{2ik_jz}{\varepsilon^2}}\widehat{\mathcal{L}}^\varepsilon(k_j,z,\boldsymbol{\kappa}_j,\boldsymbol{\kappa}_j')+e^{\frac{2ik_jz}{\varepsilon^2}}\int\int\widehat{\mathcal{R}}^\varepsilon(k_j,z,\boldsymbol{\kappa}_j,\boldsymbol{\kappa}_a)\widehat{\mathcal{L}}^\varepsilon(k_j,z,\boldsymbol{\kappa}_a,\boldsymbol{\kappa}_b)\widehat{\mathcal{R}}^\varepsilon(k_j,z,\boldsymbol{\kappa}_b,\boldsymbol{\kappa}_j')\,d\boldsymbol{\kappa}_a\,d\boldsymbol{\kappa}_b\right.$$
$$\left.+\int\widehat{\mathcal{L}}^\varepsilon(k_j,z,\boldsymbol{\kappa}_j,\boldsymbol{\kappa}_a)\widehat{\mathcal{R}}^\varepsilon(k_j,z,\boldsymbol{\kappa}_a,\boldsymbol{\kappa}_j')\,d\boldsymbol{\kappa}_a+\int\widehat{\mathcal{R}}^\varepsilon(k_j,z,\boldsymbol{\kappa}_j,\boldsymbol{\kappa}_a)\widehat{\mathcal{L}}^\varepsilon(k_j,z,\boldsymbol{\kappa}_a,\boldsymbol{\kappa}_j')\,d\boldsymbol{\kappa}_a\right\}. \tag{B.1}$$

We again apply diffusion approximations to get transport equations for the moments. We obtain from (B.1) that the limiting moments $\bar{J}_N$:

$$\bar{J}_N(z)=\lim_{\varepsilon\to0}\mathbb{E}[J_N^\varepsilon(z)],$$

solve a system of integro-differential equations

$$\frac{d}{dz}\bar{J}_N(z)=-\frac{i}{2}\left(\sum_{j=1}^N\frac{|\boldsymbol{\kappa}_j|^2+|\boldsymbol{\kappa}_j'|^2}{k_j}\right)\bar{J}_N(z)-\frac{1}{4}\sum_{j=1}^5\mathcal{J}_{N,j}(z), \tag{B.2}$$

with the initial conditions $\bar{J}_N(k_1,\ldots,k_N,\boldsymbol{\kappa}_1,\ldots,\boldsymbol{\kappa}_N,\boldsymbol{\kappa}_1',\ldots,\boldsymbol{\kappa}_N',z=0)=R_0^N\delta(\boldsymbol{\kappa}_1-\boldsymbol{\kappa}_1')\cdots\delta(\boldsymbol{\kappa}_N-\boldsymbol{\kappa}_N')$. The first term to the right-hand side of (B.2) is the contributions of the deterministic diffractive terms in (15). We next discuss the particular forms of the source terms $\mathcal{J}_{N,j}$. We consider first the interaction of the first two terms in the curly brackets in (B.1), this gives the contribution

$$\mathcal{J}_{N,1}(z)=\left(\sum_{j=1}^N k_j^2C_{2k_j}(\mathbf{0})\right)\bar{J}_N(z).$$

We consider next the interaction of the last two terms in the curly brackets in (B.1). The interaction of these terms with themselves gives the contribution

$$\mathcal{J}_{N,2}(z)=\left(\sum_{j=1}^N k_j^2C_0(\mathbf{0})\right)\bar{J}_N(z). \tag{B.3}$$

The cross interaction of these terms gives the contribution

$$\mathcal{J}_{N,3}(z)=\frac{1}{(2\pi)^d}\sum_{j=1}^N k_j^2\int\widehat{C}_0(\boldsymbol{\kappa})\bar{J}_N(z;\boldsymbol{\kappa}_j-\boldsymbol{\kappa},\boldsymbol{\kappa}_j'-\boldsymbol{\kappa})\,d\boldsymbol{\kappa}. \tag{B.4}$$

We consider finally the cross interaction of the terms in the first line of (B.1), the terms $\widehat{\mathcal{R}}^\varepsilon(k_l,z,\boldsymbol{\kappa}_l,\boldsymbol{\kappa}_l')$, with those in the curly brackets. The cross interaction with the third term in the curly brackets gives the contribution

$$\mathcal{J}_{N,4}(z)=\frac{1}{2(2\pi)^d}\sum_{j=1}^N\sum_{l=1\neq j}^N k_jk_l\int\widehat{C}_0(\boldsymbol{\kappa})\big(\bar{J}_N(z;\boldsymbol{\kappa}_j-\boldsymbol{\kappa},\boldsymbol{\kappa}_l'-\boldsymbol{\kappa})+\bar{J}_N(z;\boldsymbol{\kappa}_j-\boldsymbol{\kappa},\boldsymbol{\kappa}_l+\boldsymbol{\kappa})\big)d\boldsymbol{\kappa}. \tag{B.5}$$

The cross interaction with the fourth term in the curly brackets gives the contribution

$$\mathcal{J}_{N,5}(z)=\frac{1}{2(2\pi)^d}\sum_{j=1}^N\sum_{l=1\neq j}^N k_jk_l\int\widehat{C}_0(\boldsymbol{\kappa})\Big(\bar{J}_N(z;\boldsymbol{\kappa}_j'-\boldsymbol{\kappa},\boldsymbol{\kappa}_l-\boldsymbol{\kappa})+\bar{J}_N(z;\boldsymbol{\kappa}_j'-\boldsymbol{\kappa},\boldsymbol{\kappa}_l'+\boldsymbol{\kappa})\Big)d\boldsymbol{\kappa}. \tag{B.6}$$

We conclude that $\bar{J}_N(z)$ solves

$$\frac{d}{dz}\bar{J}_N(z) = -\frac{i}{2}\left(\sum_{j=1}^{N}\frac{|\boldsymbol{\kappa}_j|^2+|\boldsymbol{\kappa}_j'|^2}{k_j}\right)\bar{J}_N(z) - \frac{1}{4}\left(\sum_{j=1}^{N}k_j^2 C_0(\mathbf{0}) + k_j^2 C_{2k_j}(\mathbf{0})\right)\bar{J}_N(z) - \frac{1}{4(2\pi)^d}\sum_{j=1}^{N}k_j^2\int \widehat{C}_0(\boldsymbol{\kappa})\bar{J}_N(z;\boldsymbol{\kappa}_j-\boldsymbol{\kappa},\boldsymbol{\kappa}_j'-\boldsymbol{\kappa})\,d\boldsymbol{\kappa}$$
$$-\frac{1}{8(2\pi)^d}\sum_{j=1}^{N}\sum_{l=1\neq j}^{N}k_j k_l\int \widehat{C}_0(\boldsymbol{\kappa})\Big(\bar{J}_N(z;\boldsymbol{\kappa}_j-\boldsymbol{\kappa},\boldsymbol{\kappa}_l'-\boldsymbol{\kappa}) + \bar{J}_N(z;\boldsymbol{\kappa}_j'-\boldsymbol{\kappa},\boldsymbol{\kappa}_l-\boldsymbol{\kappa}) + \bar{J}_N(z;\boldsymbol{\kappa}_j-\boldsymbol{\kappa},\boldsymbol{\kappa}_l+\boldsymbol{\kappa})$$
$$+\bar{J}_N(z;\boldsymbol{\kappa}_j'-\boldsymbol{\kappa},\boldsymbol{\kappa}_l'+\boldsymbol{\kappa})\Big)d\boldsymbol{\kappa},$$

and can then verify that

$$\bar{J}_N(z) = \mathbb{E}\left[\prod_{j=1}^{N}\widehat{\mathcal{R}}(k_j,z,\boldsymbol{\kappa}_j,\boldsymbol{\kappa}_j')\right],$$

when the right-hand side expectation is taken with respect to the Itô–Schrödinger model for the reflection operator in (36).

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